San José State University

Math 161A: Applied Probability & Statistics

Conditional Probability & Independence

Prof. Guangliang Chen

Section 2.4 Conditional probability

Section 2.5 Independence

Introduction

Consider the following problem.

Example 0.1 (Toss two fair dice)**.** Let $B = \{\text{Sum=10}\}$. Find $P(B)$.

Introduction (cont'd)

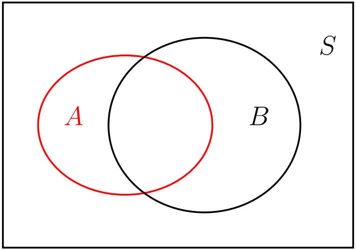
Example 0.2 (Continuation of the previous question)**.** What if we are given that the two numbers are identical (event A)?

Conditional probability

When extra information (about the result of a random phenomenon) is available, this effectively reduces the sample space.

Def 0.1. Suppose $A, B \subseteq S$ and $P(A) > 0$. The **conditional probability** of B given A (which has occurred) is defined as

$$P(B \mid \underbrace{A}_{\text{given}}) = \frac{P(A \cap B)}{P(A)}$$



Example 0.3. Consider the previous question again where the experiment was tossing two fair dice and we let $A = \{\text{Two identical numbers}\}$ and $B = \{\text{Sum=10}\}$. We already know that $P(B \mid A) = \frac{1}{6}$. Find also $P(A \mid B)$. Are they equal to each other?

Multiplication rule

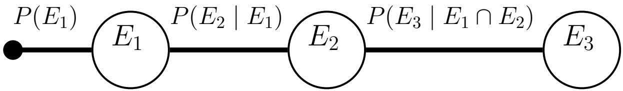
Rewriting the equation in the definition of conditional probability leads to a rule for computing the probability of several events occuring together.

Theorem 0.1. For any two events $A, B \subseteq S$ with $P(A) > 0$,

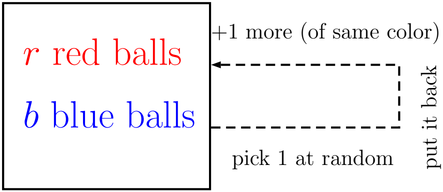
$$P(A \cap B) = P(B \mid A) \cdot P(A).$$

More generally,

$$P(E_1 \cap E_2 \cap E_3) = P(E_1) \cdot P(E_2 \mid E_1) \cdot P(E_3 \mid E_1 \cap E_2).$$



Example 0.4 (Polya's urn scheme)**.** Suppose an urn initially has r red balls and b blue balls. A ball is drawn at random and its color noted. Then it together with an extra ball of the same color (as the drawn ball) is put back into the urn. Now select a second ball at random. What is the probability that the two drawn balls are both red?



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Solution:

Example 0.5. Three cards are dealt from the top of a well-shuffled deck of 52 cards. What is the probability that they are all hearts?

Partition of a sample space

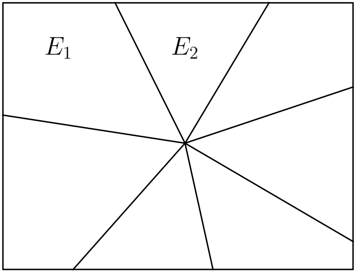
Def 0.2. A sequence of nonempty events $\{E_i\}$ are said to form a **partition** of the sample space S if they are

- **mutually exclusive**:

$$E_i \cap E_j = \emptyset \quad \text{for all } i \neq j,$$

- and **exhaustive**:

$$\cup E_i = S.$$



Example 0.6 (Toss a die once). The sample space is $S = \{1, 2, 3, 4, 5, 6\}$.

Which of the following are partitions of the sample space?

(1) $E_1 = \{1\}, \ldots, E_6 = \{6\}$

(2) $A = \{1, 3, 5\},\ A^c = \{2, 4, 6\}$

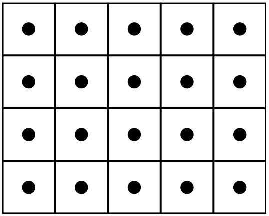
(3) $A = \{1, 2, 3\},\ B = \{4, 5\},\ C = \{6\}$

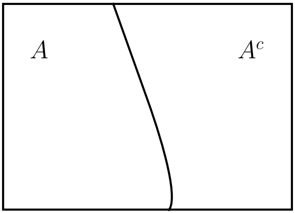
(4) $A = \{1, 3, 5\},\ B = \{2, 4\}$

(5) $A = \{1, 3, 5\},\ B = \{2, 4\},\ C = \{5, 6\}$

Remark. For any sample space S, the following are partitions:

- All the simple events: $S = \cup_{s \in S} \{s\}$.
- Any nonempty event $A \subset S$ and its complement: $S = A \cup A^c$.

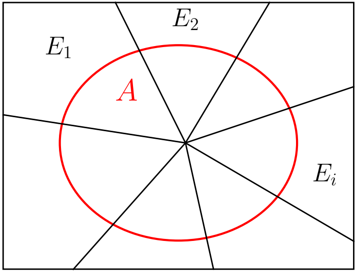




Law of total probability (LTP)

Theorem 0.2. Assume a partition of a sample space $S = E_1 \cup E_2 \cup \cdots$. For any event $A \subseteq S$, we have

$$P(A) = \sum_i P(A \cap E_i)$$
$$= \sum_i P(A \mid E_i) P(E_i).$$

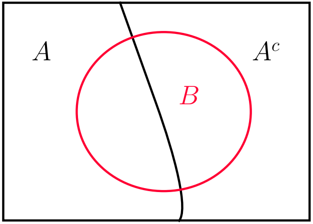


Proof. This is a direct consequence of **additivity of probability function** and the **multiplication rule**. □

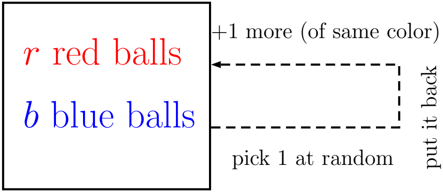
For a partition of size two: $S = A \cup A^c$, the LTP reduces to the following.

Corollary 0.3. Let $A \subset S$, with $P(A) > 0$. Then for any event $B \subseteq S$,

$$\begin{aligned} P(B) &= P(B \cap A) + P(B \cap A^c) \\ &= P(B \mid A)P(A) + P(B \mid A^c)P(A^c). \end{aligned}$$



Example 0.7 (Polya's urn scheme)**.** Suppose an urn initially has r red balls and b blue balls. A ball is drawn and its color noted. The it together with an extra ball of the same color (as the drawn ball) is added to the urn. Now select a second ball at random. What is the probability that the second drawn ball is red?



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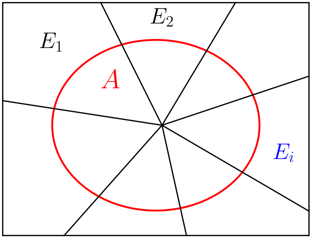
Solution:

Bayes' rule

... is a formula for computing the "posterior probabilities" $P(E_i \mid A)$.

Theorem 0.4. Suppose that the events $E_1, E_2, \ldots$ form a partition of S. Let $A \subseteq S$ be any event with $P(A) > 0$. Then, for any i,

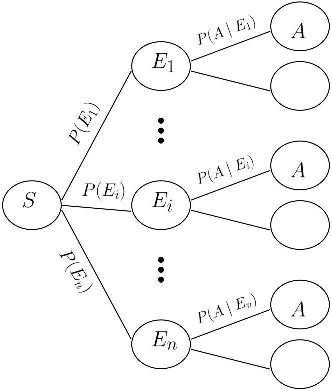
$$P(E_i \mid A) \stackrel{\text{def}}{=} \frac{P(A \cap E_i)}{P(A)} = \frac{P(A \mid E_i)P(E_i)}{\sum_j P(A|E_j)P(E_j)}$$



Interpretation of Bayes' rule:

- E_i: possible **causes or hypotheses**
- A: **evidence**
- $P(E_i)$: **prior** probabilities
- $P(A \mid E_i)$: **forward** probabilities
- $P(E_i \mid A)$: **posterior** probabilities (*after* seeing the evidence)

Law of Total Probability(LTP)

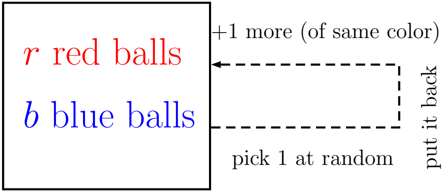


$$P(A) = P(E_1)P(A \mid E_1) + \cdots + P(E_i)P(A \mid E_i) + \cdots + P(E_n)P(A \mid E_n)$$

Bayes' Rule

$$\underbrace{P(E_i \mid A)}_{\text{Posterior Probability}} = \frac{\overbrace{P(A|E_i)}^{\text{Forward Probability}} \cdot \overbrace{P(E_i)}^{\text{Prior Probability}}}{\underbrace{P(A)}_{\text{Total Probability}}}$$

Example 0.8 (Polya's urn scheme)**.** Suppose an urn has r red balls and b blue balls. A ball is drawn and its color noted. The it together with an extra ball of the same color as the drawn ball is added to the urn. Find the probability that the first drawn ball was red given that the second ball drawn is red.



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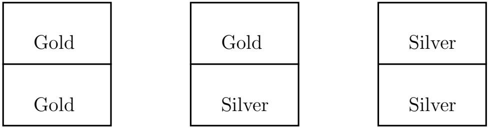
Solution:

Example 0.9. Suppose that 65% of the defendants are truly guilty. Suppose also that juries vote a guilty person innocent with probability 0.2 whereas the probability that a jury votes an innocent person guilty is 0.1. Find the probability that a defendant is convicted. What percentage of convicted defendants are truly guilty?

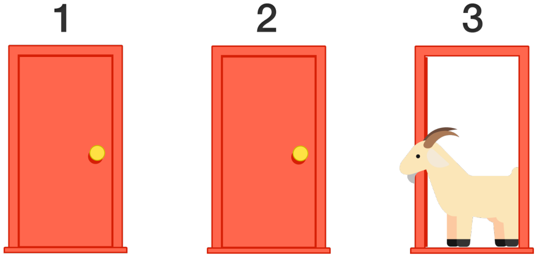
Example 0.10. Suppose there are three chests each having two drawers. One chest has a gold coin in each drawer, one chest has a gold coin in one drawer and a silver coin in the other drawer, and the third chest has a silver coin in each drawer. A chest is first picked at random and then a random drawer is opened.

(a) What is the probability that the opened drawer contains a gold coin?

(b) If the drawer contains a gold coin, what is the probability that the other drawer also contains a gold coin?



Example 0.11 (The Monty Hall problem). First watch a YouTube video at `https://www.youtube.com/watch?v=mhlc7peGlGg` and then use a tree diagram to verify the probabilities.

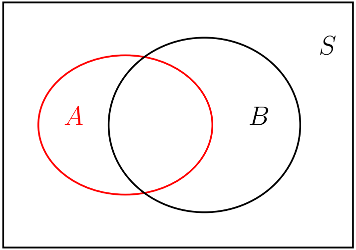


Independence

Two events are independent if the knowledge of one event occurring does not change the probability of the other occurring.

Def 0.3. Two events $A, B \subseteq S$ with $P(A) > 0$ are said to be (statistically) *independent* if

$$P(B \mid A) = P(B).$$



Example 0.12. A card is selected at random from an ordinary deck of 52. Let A denote the event that the selected card is an ace, and B a spade. Are A, B independent?

Theorem 0.5. Two events $A, B \subseteq S$ are *independent* if and only if

$$P(A \cap B) = P(A)P(B).$$

Proof. This can be easily shown by combining the multiplication rule and the definition of independence:

$$P(A \cap B) \overset{\text{always}}{=} P(A)P(B \mid A) \overset{\text{indep.}}{=} P(A)P(B)$$

Remark. If $A, B \subseteq S$ are independent events, then each of the pairs: A^c and B, A and B^c, A^c and B^c, are also independent events:

$$\begin{aligned} P(A \cap B^c) &= P(A) - P(A \cap B) = P(A) - P(A)P(B) \\ &= P(A)(1 - P(B)) = P(A)P(B^c). \end{aligned}$$

Example 0.13. Suppose we draw two cards from an ordinary deck of 52, with replacement. Find the probability that both are diamonds. What if the cards are drawn without replacement instead?

A joke on independence

A stats professor plans to travel to a conference by plane. When he passes the security check, they discover a bomb in his carry-on-baggage. Of course, he is hauled off immediately for interrogation.

"I don't understand it!" the interrogating officer exclaims. "You're an accomplished professional, a caring family man, a pillar of your parish - and now you want to destroy that all by blowing up an airplane!"

"Sorry", the professor interrupts him. "I had never intended to blow up the plane."

"So, for what reason else did you try to bring a bomb on board?!"

"Let me explain. Statistics shows that the probability of a bomb being on an airplane is 1/1,000. That's quite high if you think about it - so high that I wouldn't have any peace of mind on a flight."

"And what does this have to do with you bringing a bomb on board of a plane?"

"You see, since the probability of one bomb being on my plane is 1/1,000, the chance that there are two bombs is 1/1,000,000. If I already bring one, the chance of another bomb being around is actually 1/1,000,000, and I am much safer.

Def 0.4. A collection of events $E_1, E_2, \ldots$ are said to be (mutually) **independent** if for any subcollection $E_{i_1}, \ldots, E_{i_k}$, we have

$$P(E_{i_1} \cap \cdots \cap E_{i_k}) = P(E_{i_1}) \cdots P(E_{i_k}).$$

Remark. Three events $E_1, E_2, E_3 \subseteq S$ are independent if all four equations below are true:

$$\begin{aligned} P(E_1 \cap E_2) &= P(E_1)P(E_2) \\ P(E_1 \cap E_3) &= P(E_1)P(E_3) \\ P(E_2 \cap E_3) &= P(E_2)P(E_3) \\ P(E_1 \cap E_2 \cap E_3) &= P(E_1)P(E_2)P(E_3) \end{aligned}$$

If only the top three equations hold true, then E_1, E_2, E_3 are said to be *pairwise independent*.

In your homework you are asked to show that the top three equations do not necessarily imply the last one.

Thus, pairwise independence is weaker than mutual independence (the latter is what we typically mean by independence for three or more events).

Remark. Independence (for three or more events) is more often an assumption, or the experimental setup, given to us, which can be used to simplify calculations.

Example 0.14 (n coin tosses)**.** Consider the experiment of tossing a coin n times independently. Suppose the probability of the coin landing on heads is p. Find the probabilities that

(1) only the first k tosses are heads

(2) at least one toss is a head.

Solution: Let $H_i = \{i\text{th coin toss is a head}\}$ for each $i = 1, \ldots, n$. Then according to the question, $H_1, \ldots, H_n$ are (mutually) independent.

Summary

In this lecture we presented the following concepts:

- **Conditional probability** of one event B given another event A (with $P(A) > 0$):

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)}$$

- **Partition** of a sample space: $S = \cup E_i$ (E_i are pairwise disjoint)
- **Independence**: Two events A, B are (statistically) independent if

$$P(B \mid A) = P(B), \qquad \text{or equivalently} \quad P(A \cap B) = P(A)P(B)$$

and formulas:

- **Multiplication Rule**:

$$P(A \cap B) = P(B \mid A)P(A)$$

- **Law of Total Probability**: Given a partition $S = \cup E_i$, for any event $A \subseteq S$,

$$P(A) = \sum P(A \mid E_i)P(E_i)$$

- **Bayes' Rule**: For any event E_i in the partition,

$$\text{posterior probability} \longrightarrow \; P(E_i \mid A) = \frac{P(A \mid E_i)P(E_i)}{P(A)}$$