Worksheet 13: Dot product and orthogonality

Example 0.97. Let $\mathbf{u} = [3, 4]^T, \mathbf{v} = [-1, 1]^T$. Compute the following:

- Dot product $\mathbf{u} \cdot \mathbf{v}$
- Norms of $\mathbf{u}, \frac{1}{5}\mathbf{u}, \mathbf{v}, -2\mathbf{v}$
- Distance between $\mathbf{u}, \mathbf{v}$
- Angle between $\mathbf{u}, \mathbf{v}$

Example 0.98. The following sets of vectors of $\mathbb{R}^3$ are orthogonal sets:

- $\mathbf{e}_1 = [1, 0, 0]^T, \mathbf{e}_2 = [0, 1, 0]^T, \mathbf{e}_3 = [0, 0, 1]^T$
- $\mathbf{v}_1 = [1, 1, 1]^T, \mathbf{v}_2 = [1, -1, 0]^T, \mathbf{v}_3 = [1, 1, -2]^T$

Example 0.99. Each of the following two sets of vectors is an orthogonal basis for $\mathbb{R}^3$:

- $\mathbf{e}_1 = [1, 0, 0]^T, \mathbf{e}_2 = [0, 1, 0]^T, \mathbf{e}_3 = [0, 0, 1]^T$
- $\mathbf{v}_1 = [1, 1, 1]^T, \mathbf{v}_2 = [1, -1, 0]^T, \mathbf{v}_3 = [1, 1, -2]^T$

but the following sets are not:

- $\mathbf{v}_1 = [1, 1, 0]^T, \mathbf{v}_2 = [1, -1, 0]^T$ (only an orthogonal set)
- $\mathbf{v}_1 = [1, 0, 0]^T, \mathbf{v}_2 = [1, 1, 0]^T, \mathbf{v}_3 = [1, 1, 1]^T$ (only a basis)

Example 0.100. For the coordinate vector of $\mathbf{x} = [1, 2, 3]^T$ with respect to the orthogonal basis

$$\mathbf{v}_1 = [1, 1, 1]^T, \mathbf{v}_2 = [1, -1, 0]^T, \mathbf{v}_3 = [1, 1, -2]^T$$

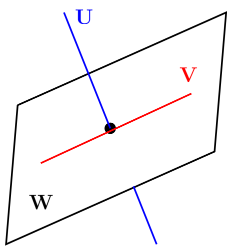
Example 0.101. Each of the following sets of vectors is an orthonormal basis for $\mathbb{R}^3$:

- $\mathbf{e}_1 = [1,0,0]^T, \mathbf{e}_2 = [0,1,0]^T, \mathbf{e}_3 = [0,0,1]^T$
- $\mathbf{v}_1 = \frac{1}{3}[1,1,1]^T, \mathbf{v}_2 = \frac{1}{\sqrt{2}}[1,-1,0]^T, \mathbf{v}_3 = \frac{1}{\sqrt{6}}[1,1,-2]^T$

Example 0.102. Find the coordinates of $\mathbf{x} = [1,2,3]^T$ with respect to the orthonormal basis $\mathbf{v}_1 = \frac{1}{\sqrt{3}}[1,1,1]^T, \mathbf{v}_2 = \frac{1}{\sqrt{2}}[1,-1,0]^T, \mathbf{v}_3 = \frac{1}{\sqrt{6}}[1,1,-2]^T$

Example 0.103. In the picture below, U, V, W are all subspaces of $\mathbb{R}^3$.

- **orthogonal subsapces**: U and V, U and W
- **orthogonal complements**: only U and W. We thus write $U = W^\perp$ and $W = U^\perp$.



Example 0.104. Consider the following matrix and its RREF

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

We have

- $\text{Row}(\mathbf{A}) = \text{span}\{[1,0,-1]^T, [0,1,2]^T\}$, and $\text{Nul}(\mathbf{A}) = \text{span}\{[1,-2,1]^T\}$.

The two subspaces are orthogonal complements of each other (inside $\mathbb{R}^3$).

On the other hand, $\text{Col}(\mathbf{A}) = \mathbb{R}^2$ and $\text{Nul}(\mathbf{A}^T) = \{\mathbf{0}\}$. The two subspaces are also orthogonal complements of each other (in $\mathbb{R}^2$).

Example 0.105. Let $\mathbf{v} = [3,4]^T$. Find the projection of $\mathbf{x} = [1,0]^T$ onto the subspace spanned by $\mathbf{v}$.

Example 0.106. Let $\mathbf{v}_1 = [1,1,0]^T, \mathbf{v}_2 = [1,-1,0]^T$. Find the projection of $\mathbf{x} = [2,3,4]^T$ onto the subspace spanned by $\mathbf{v}_1, \mathbf{v}_2$.

Worksheet 13 (cont'd): Gram-Schmidt orthogonalization process, and least squares problems

Example 0.107. Given a basis for $\mathbb{R}^2$: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, construct an orthogonal basis from it.

Example 0.108. Find an orthogonal basis for the span of $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Example 0.109. Find an orthogonal basis for the span of

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}.$$

How can we further obtain an orthonormal basis?

Example 0.110. Verify that the least squares solution of the linear system is $x = 1.92, y = 0.88$:

$$\begin{cases} x + y = 3 \\ x - y = 1 \\ 2x + 3y = 6.4 \end{cases}$$