Worksheet 1b: system of linear equations

Example 0.5. Describe the solution sets of the homogeneous system (of only one equation)

$$x_1 - 3x_2 + 2x_3 = 0$$

and the nonhomogeneous system

$$x_1 - 3x_2 + 2x_3 = 1$$

as well as their relationship.

Example 0.6. Determine if the following vectors are linearly independent:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

Example 0.7. Determine in each case if the vectors are linearly independent.

(1) $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$

(2) $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(3) $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 7 \end{bmatrix}$

(4) $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Example 0.8. Consider the transformation

$$T : \mathbb{R} \mapsto \mathbb{R}, \qquad \text{with} \quad T(x) = x^2.$$

Determine the domain, co-domain (target space), and range of T.

Example 0.9. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix}$. Then the matrix $\mathbf{A}$ may be used to define a transformation

$$T : \mathbb{R}^3 \mapsto \mathbb{R}^2, \qquad \text{with} \quad T(\mathbf{x}) = \mathbf{A}\mathbf{x}.$$

Answer the following questions:

- What are the domain and co-domain of T?
- What is the image of $\mathbf{x} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$?
- Which points in $\mathbb{R}^3$ have an image of $\mathbf{o} \in \mathbb{R}^2$?
- What is the range of T?

Example 0.10. Determine the linear transformation that maps the points $(2, 0), (1, 1)$ in $\mathbb{R}^2$ to $(-1, 0), (0, -1)$ in $\mathbb{R}^2$, respectively.

Example 0.11. Determine in each case, if the linear transformation $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ is one-to-one, or onto, or both.

- $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$
- $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix}$
- $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$