Chapter 6. Manipulator Dynamics

11-3-14

Quiz on Nov. 11 on Homework #8

Homework #8. Not collected.

Solve 6.1 (Answer partially given in the textbook). 6.12 (Answer given). 6.16.

Show how (6.32) is derived from (6.15) and (5.45).

Trace the steps taken to derive (6.36) from (6.12).

Verify the formulation of (6.42).

See the Example in Section 6.7 – Two link robot arm with simplifying assumptions. Check the vector cross multiplications at several places in the solution.

Acceleration of Rigid Body – Definition:

Acceleration of linear velocity vector V_Q in frame {B}

$$ {}^{B}\dot{V}s_Q = \frac{d}{dt}{}^{B}V_Q = \lim_{\Delta t \to 0} \frac{{}^{B}V_Q(t+\Delta t) - {}^{B}V_Q(t)}{\Delta t} \tag{6.1} $$

Acceleration of angular velocity vector ω_Q in frame {B}

$$ {}^{A}\dot{\Omega}_Q = \frac{d}{dt}{}^{A}\Omega_Q = \lim_{\Delta t \to 0} \frac{{}^{A}\Omega_Q(t+\Delta t) - {}^{A}\Omega_Q(t)}{\Delta t} \tag{6.2} $$

Linear Acceleration:

From (5.12),

$$ {}^{A}V_Q = \frac{d}{dt}({}^{A}_{B}R\,{}^{B}Q) = {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}Q \tag{6.5} $$

Differentiating (6.5) and a term for linear acceleration of the origin of {B},

$$ {}^{A}\dot{V}_Q = \frac{d}{dt}({}^{A}_{B}R\,{}^{B}V_Q) + {}^{A}\dot{\Omega}_B \times {}^{A}_{B}R\,{}^{B}Q + {}^{A}\Omega_B \times \frac{d}{dt}({}^{A}_{B}R\,{}^{B}Q) \tag{6.7} $$

$$ = ({}^{A}_{B}R\,{}^{B}\dot{V}_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}V_Q) + {}^{A}\dot{\Omega}_B \times {}^{A}_{B}R\,{}^{B}Q + {}^{A}\Omega_B \times ({}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}Q) \tag{6.8} $$

With the linear acceleration of $\{B\}_{Orig}$

$$ {}^{A}\dot{V}_Q = {}^{A}\dot{V}_{BOrg} + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}V_Q + 2\,{}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}V_Q\;{}^{A}_{B}R\,{}^{B}\dot{V}_Q + {}^{A}\dot{\Omega}_B \times {}^{A}_{B}R\,{}^{B}Q + {}^{A}\Omega_B \times ({}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}Q) \tag{6.10} $$

When ${}^{B}Q$ is constant,

$$ {}^{A}\dot{V}_Q = {}^{A}\dot{V}_{BOrg} + {}^{A}_{B}R\,{}^{B}\dot{V}_Q + {}^{A}\dot{\Omega}_B \times {}^{A}_{B}R\,{}^{B}Q + {}^{A}\Omega_B \times ({}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}Q) \tag{6.12} $$

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Angular Acceleration:

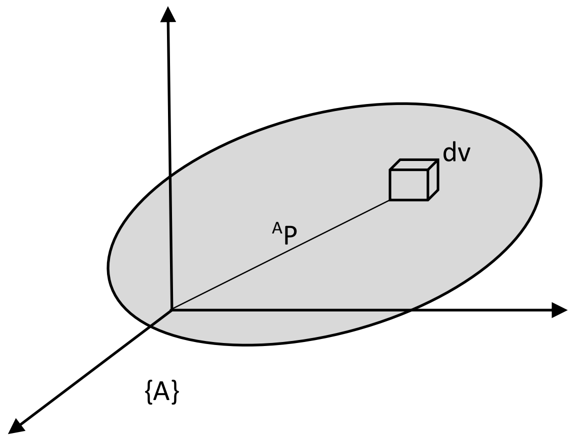
To find the angular acceleration of {C} w.r.t. {A}, differentiate

$$^{A}\Omega_{C} = {}^{A}\Omega_{B} + {}^{A}_{B}R\, {}^{B}\Omega_{C} \qquad (6.13)$$

$$^{A}\dot{\Omega}_{C} = {}^{A}\dot{\Omega}_{B} + \frac{d}{dt}({}^{A}_{B}R\, {}^{B}\Omega_{C}) = {}^{A}\dot{\Omega}_{B} + {}^{A}_{B}R\, {}^{B}\dot{\Omega}_{C} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\, {}^{B}\Omega_{C} \qquad (6.15)$$

Rigid Body Mass Distribution

Inertia tensor – Describes the distribution of the mass around the center of a rigid body.



^{A}P is the location vector of the differential volume dv.

Inertia Tensor of {A}:

$$^{A}I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$$

Mass moment of inertia:

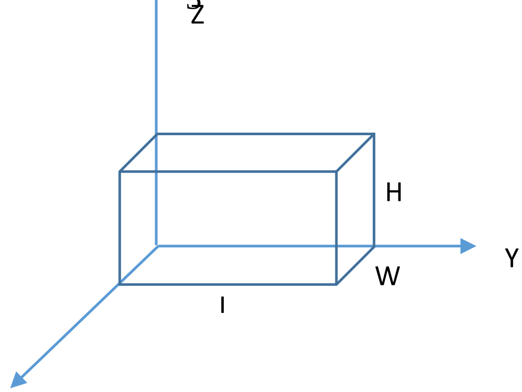
$$I_{xx} = \iiint_V (y^2 + z^2)\rho d\upsilon \quad I_{yy} = \iiint_V (x^2 + z^2)\rho d\upsilon \quad I_{zz} = \iiint_V (x^2 + y^2)\rho d\upsilon$$

$$I_{xy} = \iiint_V xy\rho d\upsilon \quad I_{xz} = \iiint_V xz\rho d\upsilon \quad I_{yz} = \iiint_V yz\rho d\upsilon$$

Example 6.1

$$I_{xx} = \int_0^h \int_0^l \int_0^w (r^2 - x^2)\rho dxdydz = \int_0^h \int_0^l \int_0^w (y^2 + z^2)\rho dxdydz = \frac{m}{3}(l^2 + h^2)$$

$$I_{xy} = \int_0^h \int_0^l xy\rho dxdydz = \frac{m}{4} wl$$



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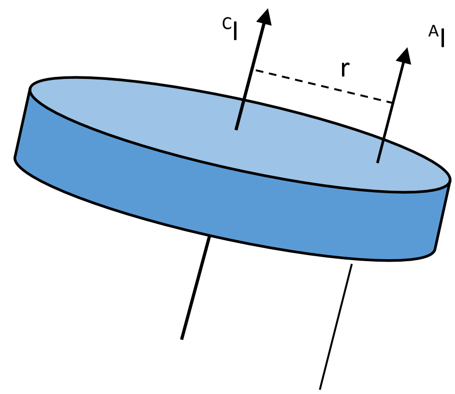
Parallel Axis Theorem:

The moment of inertia at the center of the mass is at the minimum quantity along the axis of rotation. The moment of inertia of any axis parallel to the axis of rotation is given by

$$ {}^{A}I_{zz} = {}^{C}I_{zz} + m \cdot r_c^{2} $$

where r_c = the distance from the axis in {A} to the center of the mass in {C} and m = the point mass at the center.

Inertial tensor of a mass in frame {A} w.r.t. frame {C} with its origin at the center of the mass.



$$ {}^{A}I_{zz} = {}^{C}I_{zz} + m(x_c^{2} + y_c^{2}) = {}^{C}I_{zz} + mr_c^{2} $$

$$ {}^{A}I_{xy} = {}^{C}I_{xy} - mx_c y_c $$

$$ {}^{A}P_c = \begin{bmatrix} x_c & y_c & z_c \end{bmatrix}^T $$

- Location of the center of mass in {A}.

The frame {A} has its origin at $ {}^{A}P_c = \frac{1}{2}\begin{bmatrix} w & l & h \end{bmatrix}^T $

$$ {}^{C}I_{zz} = \frac{m}{12}(w^2 + l^2) \qquad {}^{C}I_{xy} = 0 $$

$$ {}^{C}I = \begin{bmatrix} {}^{C}I_{xx} & 0 & 0 \\ 0 & {}^{C}I_{yy} & 0 \\ 0 & 0 & {}^{C}I_{zz} \end{bmatrix} $$

Example 6.2

***Newton's Equation on Force*:** $F = m\dot{v}_C$ at the center of mass

***Euler's Equation on Moment*:** $N = {}^{C}I\dot{\omega} + \omega \times {}^{C}I\omega$ at the center of mass

${}^{C}I$ = inertia tensor in frame {C} with its origin at the mass center

Newton-Euler Dynamic Equations

Derivation of angular acceleration

Forward angular velocity propagation

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$${}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$$ from (5.45)

$${}^{A}\dot{\Omega}_C = {}^{A}\dot{\Omega}_B + {}^{A}_{B}R\,{}^{B}\dot{\Omega}_C + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}\Omega_C$$ from (6.15)

(6.15')

(6.15'')

Follow the derivation of (6.32) from (6.15) and (5.45)

$${}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_i + {}^{i+1}_{i}R\,{}^{i}\omega_i \times \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} \qquad (6.32)$$

For prismatic joints, ${}^{i}\omega_i = \dot{\theta}_i = 0$, so

$${}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_i$$

Derivation of linear acceleration

From (6.12) and by taking similar steps to derive the angular acceleration,

$${}^{A}V_Q = \frac{d}{dt}({}^{A}_{B}R\,{}^{B}Q) = {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}Q$$

$${}^{A}\dot{V}_Q = {}^{A}\dot{V}_{BOrg} + {}^{A}_{B}R\,{}^{B}\dot{V}_Q + {}^{A}\dot{\Omega}_B \times {}^{A}_{B}R\,{}^{B}Q + {}^{A}\Omega_B \times ({}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}Q)$$

Setting {A}={i+1} and {B}={i} and factoring out ${}^{i+1}_{i}R$,

$${}^{i+1}\dot{\upsilon}_{i+1} = {}^{i+1}_{i}R[{}^{i}\dot{\upsilon}_i + {}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times ({}^{i}\omega_i \times {}^{i}P_{i+1})] \qquad (6.34)$$

For *prismatic joints*, add two more terms to (6.34) per (6.10)

$${}^{i+1}\dot{\upsilon}_{i+1} = {}^{i+1}_{i}R[{}^{i}\dot{\upsilon}_i + {}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times ({}^{i}\omega_i \times {}^{i}P_{i+1})] + 2\,{}^{i+1}\omega_{i+1} \times \dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} \qquad (6.35)$$

Linear acceleration of the center of mass, from (6.12)

Trace the steps taken in applying (6.12),

$${}^{i}\dot{\upsilon}_{Ci} = {}^{i}\dot{\omega}_i \times {}^{i}P_{Ci} + {}^{i}\omega_i \times ({}^{i}\omega_i \times {}^{i}P_{Ci}) + {}^{i}\dot{v}_i \qquad (6.36)$$

The inertial force and torque acting at the center of the mass:

From (6.32) and (6.36)

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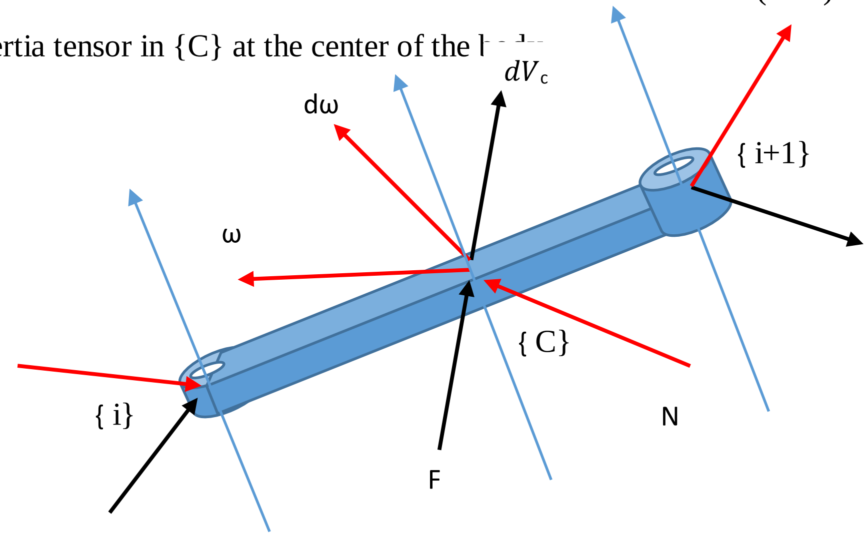
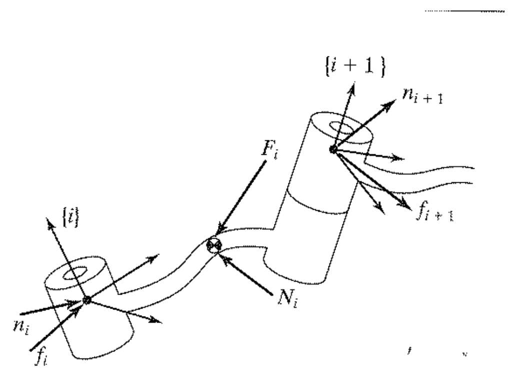
$$F_i = m\dot{\upsilon}_{C_i}$$
$$N_i = {}^{C_i}I\dot{\omega}_i + \omega_i \times {}^{C_i}I\omega_i$$
(6.37)

where 3x3 ${}^{C_i}I$ is the inertia tensor in {C} at the center of the b...



H7Backward Iteration for Joint Forces and Torque



Force and torque balance equations at the center of mass of link i:

$${}^iF_i = {}^if_i - {}^i_{i+1}R\,{}^{i+1}f_{i+1} \tag{6.38}$$

$${}^iN_i = {}^in_i - {}^in_{i+1} + ({}^iP_i - {}^iP_{Ci}) \times {}^if_i - ({}^iP_{i+1} - {}^iP_{Ci}) \times {}^if_{i+1} \tag{6.39}$$

$${}^iP_i = 0$$

Rearranging the equations and adding rotations;

$${}^if_i = {}^i_{i+1}R\,{}^{i+1}f_{i+1} + {}^iF_i \tag{6.41}$$

$${}^in_i = {}^iN_i + {}^i_{i+1}R\,{}^{i+1}n_{i+1} + {}^iP_{Ci} \times {}^iF_i + {}^iP_{i+1} \times {}^i_{i+1}R\,{}^{i+1}f_{i+1} \tag{6.42}$$

Finally, the joint torque is the Z component of the vector representing the inertial torque:

$$\tau_i = {}^in_i^T\,{}^i\hat{Z}_i \tag{6.43}$$

For prismatic joints, using τ to denote force:

$$\tau_i = {}^if_i^T\,{}^i\hat{Z}_i \tag{6.44}$$

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Forward and backward iterations: Eq (6.45)-(6.53)

Forward - Link velocities and accelerations via the Newton-Euler (6.31)-(6.37).

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} \qquad (6.45)$$

$$ {}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_i + {}^{i+1}_{i}R\,{}^{i}\omega_i \times \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} \qquad (6.46)$$

$$ {}^{i+1}\dot{\upsilon}_{i+1} = {}^{i+1}_{i}R[{}^{i}\dot{\upsilon}_i + {}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times ({}^{i}\omega_i \times {}^{i}P_{i+1})] \qquad (6.47)$$

$$ {}^{i+1}\dot{\upsilon}_{C_{i+1}} = {}^{i}\dot{\omega}_i \times {}^{i+1}P_{i+1} + {}^{i+1}\omega_{i+1} \times ({}^{i+1}\omega_{i+1} \times {}^{i+1}P_{i+1}) + {}^{i+1}\dot{\upsilon}_{i+1} \qquad (6.48)$$

$$ {}^{i+1}F_{i+1} = m_{i+1}\,{}^{i+1}\dot{\upsilon}_{C_{i+1}} $$
$$ {}^{i+1}N_{i+1} = {}^{C_{i+1}}I_{i+1}\,{}^{i+1}\dot{\omega}_{i+1} + \qquad (6.49, 6.50)$$
$$ {}^{i+1}N_{i+1} = {}^{C_{i+1}}I_{i+1}\,{}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{i+1}I_{i+1}\,{}^{i+1}\omega_{i+1} $$

Backward - Find joint forces and torques via (6.38)-(6.44).

$$ {}^{i}f_i = {}^{i}I\,{}_{i+1}^{i}R\,{}^{i+1}f_{i+1} + F_i \qquad (6.51)$$

$$ {}^{i}n_i = {}^{i}N_i + {}_{i+1}^{i}R\,{}^{i+1}n_{i+1} + {}^{i}P_i \times {}^{i}F_i + {}^{i}P_{i+1} \times {}_{i+1}^{i}R\,{}^{i+1}f_{i+1} \qquad (6.52)$$

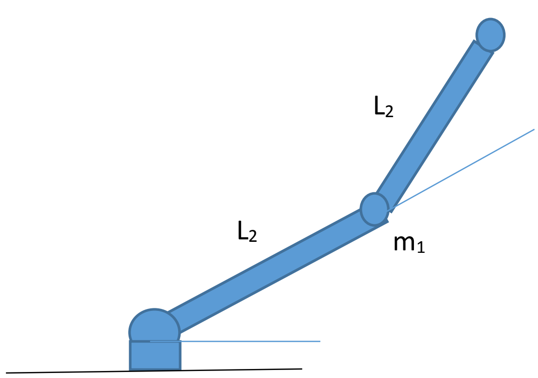
$$ \tau_i = {}^{i}n_i^{T}\,{}^{i}\hat{Z}_i \qquad (6.53)$$

See the Example in Section 6.7 – Simplified two link robot arm.

Check the vector cross multiplications at several places in the solution.

$$ \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} $$

$$ \mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k} $$



For Link 1:

$$^1\omega_1 = \omega_0 + \dot{\theta}_1{}^1\hat{Z}_1 = \begin{bmatrix} 0 & 0 & \dot{\theta}_1 \end{bmatrix}^T$$

$$^1\dot{\omega}_1 = \ddot{\theta}_1{}^1\hat{Z}_1 = \begin{bmatrix} 0 & 0 & \ddot{\theta}_1 \end{bmatrix}^T$$

$$^1\dot{\upsilon}_1 = {}^1_0R^i\dot{\upsilon}_i = \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix} = \begin{bmatrix} gs_1 \\ gc_1 \\ 0 \end{bmatrix}$$

$$^1\dot{\upsilon}_{C^1} = {}^1\dot{\omega}_1 \times {}^1P_{C^1} + {}^1\omega_1 \times ({}^1\omega_1 \times {}^1P_{C^1}) + {}^1\dot{\upsilon}_1 = \begin{bmatrix} i & j & k \\ 0 & 0 & \ddot{\theta}_1 \\ l_1 & 0 & 0 \end{bmatrix} + {}^1\omega_1 \times \begin{bmatrix} i & j & k \\ 0 & 0 & \dot{\theta}_1 \\ l_1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} gs_1 \\ gc_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -l_1\dot{\theta}_1^2 + gs_1 \\ l_1\ddot{\theta}_1 + gs_1 \\ 0 \end{bmatrix}$$

$$^1F_1 = m_1{}^1\dot{\upsilon}_1 = m_1\begin{bmatrix} -l_1\dot{\theta}_1^2 + gs_1 \\ l_1\ddot{\theta}_1 + gs_1 \\ 0 \end{bmatrix}$$

$$^1N_1 = {}^{C_{i+1}}I_{i+1}{}^1\dot{\omega}_1 + {}^1\omega_1 \times {}^{i+1}I_{i+1}{}^1\omega_1 = 0\cdot{}^1\dot{\omega}_1 + {}^1\omega_1 \times 0\cdot{}^1\omega_1 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$$

For Link 2:

$$^2\omega_2 = {}^2_1R^1\omega_1 + \dot{\theta}_2{}^2\hat{Z}_2 = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}$$

$$^2\dot{\omega}_2 = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 + \ddot{\theta}_2 \end{bmatrix}$$

$$^2\dot{v}_2 = {}^2_1R[{}^1\dot{\upsilon}_1 + {}^1\dot{\omega}_1 \times {}^1P_2 + {}^1\omega_1 \times ({}^1\omega_1 \times {}^2P_2)] = {}^2_1R^2\dot{v}_{C1} = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -l_1\dot{\theta}_1^2 + gs_1 \\ l_1\ddot{\theta}_1 + gs_1 \\ 0 \end{bmatrix}$$

$$^2\dot{\upsilon}_{C^2} = {}^1\dot{\omega}_1 \times {}^2P_{C2} + {}^2\omega_2 \times ({}^2\omega_2 \times {}^2P_{C2}) + {}^2\dot{\upsilon}_2 = \begin{bmatrix} i & j & k \\ 0 & 0 & \ddot{\theta}_1 \\ l_2 & 0 & 0 \end{bmatrix} + {}^2\omega_2 \times \begin{bmatrix} i & j & k \\ 0 & 0 & \dot{\theta}_1 + \dot{\theta}_2 \\ l_2 & 0 & 0 \end{bmatrix} + {}^2\dot{\upsilon}_2$$

$$^2F_2 = m_2{}^2\dot{\upsilon}_{C^2}$$

$$^2N_2 = {}^{C_2}I_2{}^2\dot{\omega}_2 + {}^2\omega_2 \times {}^2I_2{}^2\omega_2 = [0]^2\dot{\omega}_2 + {}^2\omega_2 \times [0]^2\omega_2 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$$

The backward calculations of $^2f_2.{}^2n_2, {}^1f_1.{}^1n_1$ for *Links 2 and 1 can be carried out similarly.*

The joint torques τ_1, τ_2 are the Z components of $.{}^1n_1, {}^2n_2$.

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Dynamic Equations – General Structure

State Space equation:

$$\tau = M(\Theta)\ddot{\Theta} + V(\Theta,\dot{\Theta}) + G(\Theta) \tag{6.59}$$

where,

$M(\Theta)$ = n x n mass matrix for the terms containing $\ddot{\theta}_i, i = 1..n$

$V(\Theta,\dot{\Theta})$= n x 1 vector containing centrifugal ($\dot{\theta}_i^2$) and Coriolis ($\dot{\theta}_i\dot{\theta}_j$) terms

$G(\Theta)$= n x 1 vector containing gravity (g) terms.

Configuration Space equation:

$$\tau = M(\Theta)\ddot{\Theta} + B(\Theta)\left[\dot{\Theta}\dot{\Theta}\right] + C(\Theta)\left[\dot{\Theta}^2\right] + G(\Theta) \tag{6.63}$$

where,

$B_x(\Theta)$ = a matrix of Coriolis coefficients

$C_x(\Theta)$ = matrix of centrifugal coefficients

$\left[\dot{\Theta}\dot{\Theta}\right] = \left[\dot{\theta}_1\dot{\theta}_2 \quad \dot{\theta}_1\dot{\theta}_3 \quad ... \quad \dot{\theta}_{n-1}\dot{\theta}_n\right]$, a vector of joint velocity products (6.64)

$\left[\dot{\Theta}^2\right] = \left[\dot{\theta}_1{}^2 \quad \dot{\theta}_1{}^2 \quad ... \quad \dot{\theta}_n{}^2\right]$, a matrix of centrifugal coefficients (6.65)

The 2 link manipulator example:

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = \begin{bmatrix}\mu_{11} & \mu_{12}\\ \mu_{21} & \mu_{22}\end{bmatrix}\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix} + \begin{bmatrix}b_1\\ b_2\end{bmatrix}\left[\dot{\theta}_1\,\dot{\theta}_2\right] + \begin{bmatrix}c_{11} & c_{12}\\ c_{21} & c_{22}\end{bmatrix}\begin{bmatrix}\dot{\theta}_1^2\\ \dot{\theta}_2^2\end{bmatrix} + G(\Theta)$$

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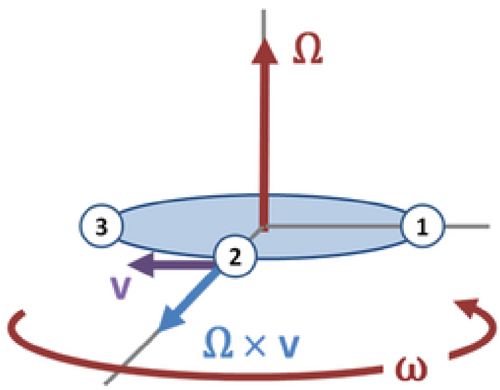
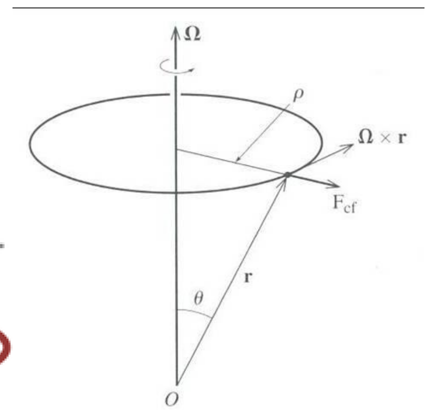
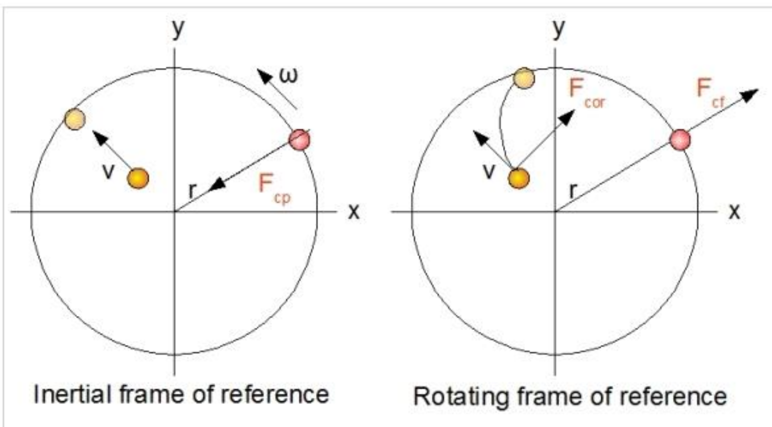
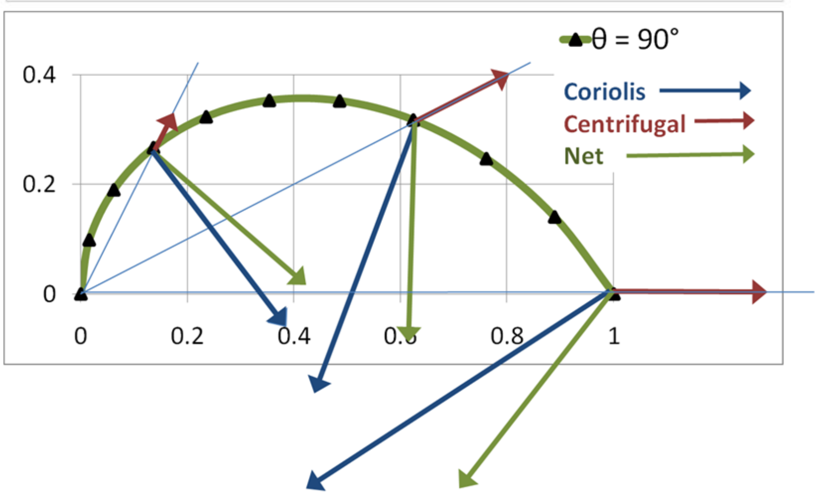
Internet Downloads:

The Coriolis effect is a deflection of moving objects when they are viewed in a rotating reference frame. In a reference frame with clockwise rotation, the deflection is to the left of the motion of the object; in one with counter-clockwise rotation, the deflection is to the right. ----Wikipedia ------









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Lagrangian Dynamic Formulation

Quadratic form of manipulator kinetic energy, analogous to k = ½ mv^2

Kinetic energy: $$k_i = \frac{1}{2} m_i \upsilon_{Ci}^T \upsilon_{Ci} + \frac{1}{2} \omega_i^T \, {}^{Ci}I_i \, \omega_i \qquad (6.69)$$

$$k = \sum k_i$$

$$k(\Theta, \ddot{\Theta}) = \frac{1}{2} \dot{\Theta}^T M(\Theta) \dot{\Theta} \qquad (6.71)$$

← in vector form

Potential energy: $$u_i = -m_i \, {}^0 g^T \, {}^o P_{Ci} + u_{ref\,i} \qquad (6.73)$$

$$u = \sum u_i$$

Lagrangian The difference between the kinetic energy and the potential energy of a body:

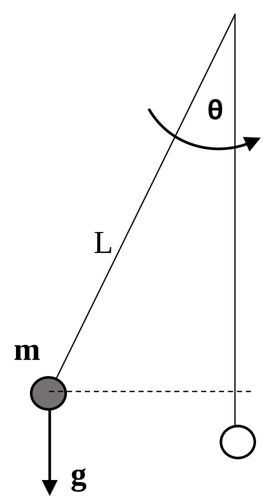
$$L(\Theta, \dot{\Theta}) = k(\Theta, \dot{\Theta}) - u(\Theta) \qquad (6.75)$$

Lagrangian Equation of Motion (*as applied to robot arms*) derived from Newton's Law,

$$\frac{d}{dt}\frac{\delta L}{\delta \dot{\Theta}} - \frac{\delta L}{\delta \Theta} = \tau \qquad (6.77)$$

$$\frac{d}{dt}\frac{\delta k}{\delta \dot{\Theta}} - \frac{\delta k}{\delta \Theta} + \frac{\delta u}{\delta \Theta} = \tau$$

Example application of Lagrangian equation – Pendulum



Kinetic energy $K = \frac{1}{2} m (l\omega)^2$

Potential energy $U = mgl(1 - \cos\theta)$

Lagrangian $L = K - U = \frac{1}{2} m (l\omega)^2 - mgl(1 - \cos\theta)$

$$\frac{\delta L}{\delta \theta} = -mgl \sin\theta \qquad \frac{\delta L}{\delta \omega} = ml^2 \omega \qquad \frac{d}{dt}\frac{\delta L}{\delta \omega} = ml^2 \dot{\omega}$$

Lagrangian equation: $\frac{d}{dt}\frac{\delta L}{\delta \omega} - \frac{\delta L}{\delta \theta} = ml^2 \dot{\omega} - mgl \sin\theta = 0$

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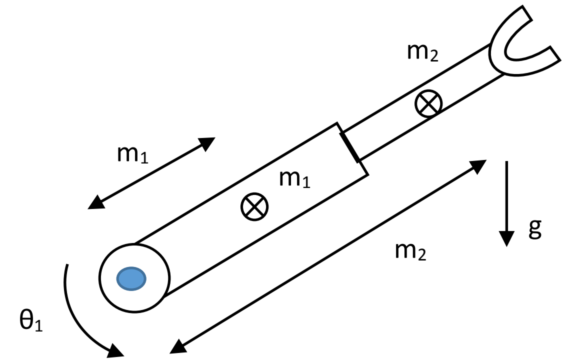
Example 6.5 – An RP manipulator



Kinetic energy: $k_1 = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} I_{zz1} \dot{\theta}_1^2$ $k_2 = \frac{1}{2} m_2 (d_2^2 \dot{\theta}_1^2 + \dot{d}_2^2) + \frac{1}{2} I_{zz2} \dot{\theta}_1^2$

$$k(\Theta, \ddot{\Theta}) = k_1 + k_2$$

Potential energy: $u_1 = m_1 l_1 g \sin\theta_1 + m_1 l_1 g$ $u_2 = m_2 d_2 g \sin\theta_1 + m_2 d_{2\max} g$

$$u(\Theta) = u_1 + u_2$$

$$\frac{\delta k}{\delta \dot{\Theta}} = \begin{bmatrix} (m_1 l_1^2 + I_{zz1} + I_{zz2} + m_2 d_2^2)\dot{\theta}_1 \\ m_2 \dot{d}_2 \end{bmatrix}$$

$$\frac{d}{dt}\frac{\delta k}{\delta \dot{\Theta}} = \begin{bmatrix} (m_1 l_1^2 + I_{zz1} + I_{zz2} + m_2 d_2^2)\ddot{\theta}_1 + 2 m_2 \dot{\theta}_1 d_2 \dot{d}_2 \\ m_2 \ddot{d}_2 \end{bmatrix}$$

$$\frac{\delta k}{\delta \Theta} = \begin{bmatrix} 0 \\ m_2 d_2 \dot{\theta}_1^2 \end{bmatrix}$$

$$\frac{\delta u}{\delta \Theta} = \begin{bmatrix} g(m_1 l_1 + m_2 d_2)\cos\theta_1 \\ g m_2 \sin\theta_1 \end{bmatrix},$$

Lagrangian equation:

$$\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \frac{d}{dt}\frac{\delta k}{\delta \dot{\Theta}} - \frac{\delta k}{\delta \Theta} + \frac{\delta u}{\delta \Theta}$$

From $\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$, extract

$M(\Theta) =$ 2 x 2 matrix of $\ddot{\theta}_1, \ddot{d}_2$ terms – Angular acceleration

$V(\Theta, \dot{\Theta}) =$ 2 x 1 vector of $\dot{\theta}_1^2, \dot{\theta}_1, \dot{d}_2^2, \dot{d}_2, \dot{\theta}_1 \dot{d}_2$ terms – Centrifugal and Coriolis

$G(\Theta) =$ 2 x 1 vector of g terms - Gravity

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Formulation in Cartesian Space

From the *Joint space* equation : $\tau = M(\Theta)\ddot{\Theta} + V\left(\Theta, \dot{\Theta}\right) + G(\Theta)$

The equivalent *Cartesian space* equation for Force-Torque vector F:

$$F = M_x(\Theta)\ddot{X} + V_x\left(\Theta, \dot{\Theta}\right) + G_x(\Theta) \qquad (6.91)$$

From $\tau = J^T(\Theta)F \rightarrow F = J^{-T}(\Theta)\tau$

$$F = J^{-T}[M(\Theta)\ddot{\Theta} + V\left(\Theta, \dot{\Theta}\right) + G(\Theta)] \qquad (6.94)$$

Defining a Jacobian relating the Joint space equation to the Cartesian space equation

$$\dot{X} = J\dot{\Theta}$$

$$\rightarrow \ddot{X} = \dot{J}\dot{\Theta} + J\ddot{\Theta}$$

$$\rightarrow \ddot{\Theta} = J^{-1}(\ddot{X} - \dot{J}\dot{\Theta}) \qquad (6.97)$$

Substituting (6.97) into (6.94)

$$F = J^{-T}[M(\Theta)J^{-1}(\ddot{X} - \dot{J}\dot{\Theta})) + V\left(\Theta, \dot{\Theta}\right) + G(\Theta)] \qquad (6.98)$$

Then, the relationship between the Cartesian expression and the Joint expression in light of (6.91) is:

$$M_x(\Theta) = J^{-T}M(\Theta)J^{-1}(\Theta)$$

$$V_x\left(\Theta, \dot{\Theta}\right) = J^{-T}[V(\Theta, \dot{\Theta}) - M(\Theta)J^{-1}\dot{J}\dot{\Theta}] \qquad (6.99)$$

$$G_x(\Theta) = J^{-T}G(\Theta)$$

Example 6.6: Derive the Cartesian space dynamics equations for the two link RR robot,

Given $J(\Theta) = \begin{bmatrix} l_1 s_2 & 0 \\ l_1 c_2 & l_2 \end{bmatrix}$.from (5.55),

find $J^{-1}(\Theta) = \dfrac{1}{l_1 l_2 s_2}\begin{bmatrix} l_2 & 0 \\ -l_1 c_2 - l_2 & l_1 s_2 \end{bmatrix}$, $\dot{J}(\Theta) = \begin{bmatrix} l_1 c_2 \dot{\theta}_2 & 0 \\ -l_1 s_2 \dot{\theta}_2 & 0 \end{bmatrix}$

Then, derive $M_x(\Theta), V_x\left(\Theta, \dot{\Theta}\right), G_x(\Theta)$ from (6.99), (6.60),(6.61), and (6.62)

Cartesian configuration space torque

$$\tau = J^T(\Theta)[M_x(\Theta)\ddot{X} + V_x\left(\Theta, \dot{\Theta}\right) + G_x(\Theta)] \qquad (6.104)$$

or dividing Vx terms into Coriolis and Centrifugal terms,

$$\tau = J^T(\Theta)M(\Theta)\ddot{X} + B_x(\Theta)\left[\dot{\Theta}\dot{\Theta}\right] + C_x(\Theta)\left[\dot{\Theta}^2\right] + G_x(\Theta) \qquad (6.105)$$

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Friction

Friction force $F(\Theta,\dot{\Theta})$ may be added to (6.59) or (6.104) to account for the effect of friction on

Dynamic Simulation

Numerical integration method is used to solve the acceleration problem of the manipulator as torque is applied to the joints,

From (6.104), solve for

$$\ddot{\Theta} = M^{-1}(\Theta)\left[\tau - V(\Theta,\dot{\Theta}) - G(\Theta) - F(\Theta,\dot{\Theta})\right] \tag{6.115}$$

$\dot{\Theta}(t+\Delta t) = \dot{\Theta}(t) + \ddot{\Theta}(\Delta t)$, *analogous to* $V(t)=V_0 + at$ and

$$\Theta(t+\Delta t) = \Theta(t) + \dot{\Theta}(t)\Delta t + \frac{1}{2}\ddot{\Theta}(\Delta t)\Delta t^2, \text{ analogous to } S(t)=S_0 + V_0(t) + ½\, at^2 \tag{6.117}$$

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