Chapter 5 Jacobians: Velocities and Static Forces

***HW #6. Due 10/15/14. 5.1, 5.7, 5.8, 5.10, 5.16.** Help available on Monday 10/16.*

Velocity of time varying position vector Q in frame {B}, by definition

$$ {}^{B}V_{Q} = \frac{d}{dt}{}^{B}Q = \lim_{\Delta t \to 0} \frac{Q(t+\Delta t) - Q(t)}{\Delta t} \quad (5.1)$$

The same reflected on frame {A}

$$ {}^{A}({}^{B}V_{Q}) = {}^{A}_{B}R\,{}^{B}V_{Q} \quad (5.4)$$

Velocity of the origin of a translating frame {B} in terms of {A}

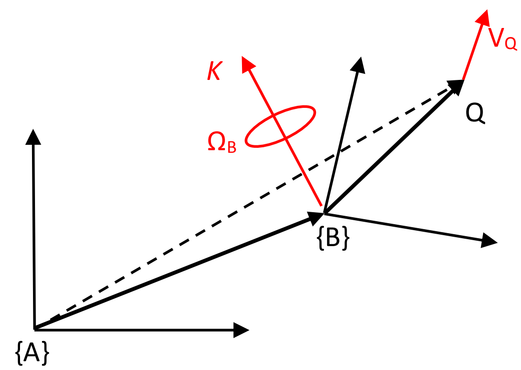
$$ {}^{A}\upsilon_{BOrg} = {}^{A}V_{BOrg} \; (\upsilon \text{ to denote the magnitude of } V) \quad (5.5)$$

Angular velocity of a rotating frame {B} in terms of {A}

$$ {}^{A}\omega_{B} = {}^{A}\Omega_{B} \quad (\omega \text{ to denote the magnitude of } \Omega) \quad (5.6)$$

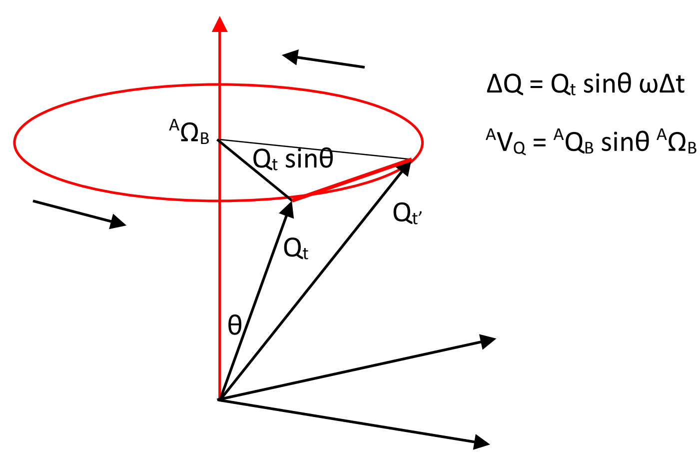
Linear velocity of Q in frame {B} which is moving relative to {A}

$$ {}^{A}V_{Q} = {}^{A}V_{BOrg} + {}^{A}_{B}R\,{}^{B}V_{Q} \quad (5.7)$$



Rotational velocity of frame {B} with respect to {A}, ${}^{A}\Omega_{B}$, applied to ${}^{A}Q_{B}$,

$$ {}^{A}V_{Q} = {}^{A}\Omega_{B} \times {}^{A}Q \text{, a vector cross product} \quad (5.10)$$



With Q_B moving at velocity ${}^{B}V_{Q}$ in {B} and frame {B} moving at ${}^{A}V_{BOrg}$ with respect to {A}, the linear and rotational velocity of Q in moving and turning frame {B} with respect to {A},

$$ {}^{A}V_{Q} = {}^{A}V_{BOrg} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}Q \quad (5.13)$$

Property of orthonormal rotation matrix for velocity analysis

Taking a derivative of $RR^T = I_n$

$$\dot{R}R^T + R\dot{R}^T = \dot{R}R^T + (\dot{R}R^T)^T = 0_n \tag{5.16}$$

So, $\dot{R}R^T = \dot{R}R^{-1}$ is a skew-symmetric matrix (in the form of $S + S^{-1} = 0$).

Velocity of vector P due to rotating frame

$$^{A}V_P = {}^{A}_{B}\dot{R}\,{}^{B}P \tag{5.22}$$

$$^{A}V_P = {}^{A}_{B}\dot{R}\,{}^{A}_{B}R^{-1}\,{}^{A}P$$

$$^{A}\boldsymbol{V}_P = {}^{A}_{B}\boldsymbol{S}\,{}^{A}\boldsymbol{P}\ , \tag{5.24}$$

S is a skew-symmetric matrix ($S+S^{-1}=0$)

Skew-symmetric matrices and vector cross product

If, $S = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix}$, $\Omega = \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix}$, and P= $\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$ and then

$$SP = \Omega \times P = \begin{bmatrix} -\Omega_z P_y + \Omega_y P_z \\ \Omega_z P_x - \Omega_x P_z \\ -\Omega_y P_x + \Omega_x P_y \end{bmatrix} \tag{5.27}$$

Then, from (5.24)

$$^{A}\boldsymbol{V}_P = {}^{A}\boldsymbol{\Omega}_B \times {}^{A}\boldsymbol{P} \tag{5.28}$$

Physical Interpretation of Angular Velocity Vector Ω

$$\dot{R} = \lim_{\Delta t \to 0} \frac{R(t+\Delta t) - R(t)}{\Delta t} \tag{5.29}$$

$$R(t+\Delta t) = R_K(\Delta\theta)R(t) \tag{5.30}$$

→ *During Δt, R rotates by Δθ about vector K, an equivalent axis of rotation.*

From (2.80) and $\sin(x) \approx x, \cos(x) \approx 1$ for a small value of x according to the Taylor expansion,

$$R_K(\Delta\theta) \approx \begin{bmatrix} 1 & -k_z\Delta\theta & k_y\Delta\theta \\ k_z\Delta\theta & 1 & -k_x\Delta\theta \\ -k_y\Delta\theta & k_x\Delta\theta & 1 \end{bmatrix} \tag{5.33}$$

Substituting (5.33) into (5.30), and (5.30) into (5.29), and taking the limit

$$\dot{R} = \begin{bmatrix} 0 & -k_z\dot{\theta} & k_y\dot{\theta} \\ k_z\dot{\theta} & 0 & -k_x\dot{\theta} \\ -k_y\dot{\theta} & k_x\dot{\theta} & 0 \end{bmatrix} R(t) \tag{5.35}$$

Transposing *R(t)* to the LHS, and recognizing the RHS as a skew symmetric matrix,

$$\dot{R}R^{-1} = \dot{R}R^T = \begin{bmatrix} 0 & -k_z\dot{\theta} & k_y\dot{\theta} \\ k_z\dot{\theta} & 0 & -k_x\dot{\theta} \\ -k_y\dot{\theta} & k_x\dot{\theta} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} \tag{5.36}$$

where $\Omega = \begin{bmatrix}\Omega_x & \Omega_y & \Omega_z\end{bmatrix}^T = \begin{bmatrix}k_x\dot{\theta} & k_y\dot{\theta} & k_z\dot{\theta}\end{bmatrix}^T = \dot{\theta}\cdot\hat{K}$ (5.37)

Ω = angular velocity vector,

K = instantaneous axis of rotation

Another Interpretation of angular velocity based on simultaneous Euler rotation Z-Y-Z,

$\dot{\Theta}_{Z'Y'Z'} = \begin{bmatrix}\dot{\alpha} & \dot{\beta} & \dot{\gamma}\end{bmatrix}^T$ `

From (2.73) and (5.36), find $\dot{R}R$, a 3x3 matrix.

$$\dot{R}R^{-1} = \dot{R}R^T = \begin{bmatrix} \dot{r}_{11} & \dot{r}_{12} & \dot{r}_{13} \\ \dot{r}_{21} & \dot{r}_{22} & \dot{r}_{23} \\ \dot{r}_{31} & \dot{r}_{32} & \dot{r}_{33} \end{bmatrix}\begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix}, \text{ where } R = R_{Z'Y'Z'}(\alpha,\beta,\gamma) \text{ from (2.72)}$$

Equating the elements (3,2), (1,3) and (2,1) to find the components in (5.37),

$$\begin{aligned} \Omega_x &= \dot{r}_{31}r_{21} + \dot{r}_{32}r_{22} + \dot{r}_{33}r_{23} \\ \Omega_y &= \dot{r}_{11}r_{31} + \dot{r}_{12}r_{32} + \dot{r}_{13}r_{33} \\ \Omega_z &= \dot{r}_{21}r_{11} + \dot{r}_{22}r_{12} + \dot{r}_{23}r_{13} \end{aligned} \tag{5.40}$$

In a general form,

$$\Omega = [E_{Z'Y'Z'}(\Theta_{Z'Y'Z'})]\dot{\Theta}_{Z'Y'Z'} \tag{5.41}$$

$E_{Z'Y'Z'}(.)$ = a Jacobian operator relating angle-set velocity vector $\dot{\Theta}_{Z'Y'Z'} = [\dot{\theta}_x, \dot{\theta}_y, \dot{\theta}_z]_T$ = to angular velocity $\Omega = [\Omega_x, \Omega_y, \Omega_z]$.

Velocity Propagation for revolute joints – More equations!

The angular velocity of link i+1 rotating about its Z axis, projected onto link i which also rotates,

$${}^{i}\omega_{i+1} = {}^{i}\omega_i + {}_{i+1}^{i}R\dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} = {}^{i}\omega_i + {}_{i+1}^{i}R\cdot{}^{i+1}\begin{bmatrix}0 & 0 & \dot{\theta}_{i+1}\end{bmatrix}^T \tag{5.44}$$

Multiplying both sides by ${}_{i}^{i+1}R$

$${}_{i}^{i+1}R\,{}^{i}\omega_{i+1} = {}^{i+1}\omega_{i+1} = {}_{i}^{i+1}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} \tag{5.45}$$

Linear velocity of frame {i+1},

$$ {}^{i}\upsilon_{i+1} = {}^{i}\upsilon_i + {}^{i}\omega_i \times {}^{i}P_{i+1} \tag{5.46}$$

Multiplying both sides by ${}^{i+1}_{i}R$

$$ {}^{i+1}\upsilon_{i+1} = {}^{i+1}_{i}R({}^{i}\upsilon_{i+1} + {}^{i}\omega_i \times {}^{i}P_{i+1}) \tag{5.47}$$

For prismatic joint

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i $$

$$ {}^{i+1}\upsilon_{i+1} = {}^{i+1}_{i}R({}^{i}\upsilon_i + {}^{i}\omega_i \times {}^{i}P_{i+1}) + \dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} \tag{5.48}$$

Example 5.3: - RRR robot with frozen joint 3.

${}^{i}\upsilon_i, {}^{i}\omega_i, {}^{i}P_{i+1}$ are 3x1 vectors. The vector cross products in (5.47) ${}^{i}\omega_i \times {}^{i}P_{i+1}$ are shown below.

$$ {}^{1}\omega_1 \times {}^{1}P_2 = \begin{bmatrix} i & j & k \\ 0 & 0 & \dot{\theta}_1 \\ l_1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ l_1\dot{\theta}_1 \\ 0 \end{bmatrix} \qquad {}^{2}\omega_2 \times {}^{2}P_3 = \begin{bmatrix} i & j & k \\ 0 & 0 & \dot{\theta}_1 + \dot{\theta}_2 \\ l_2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ l_2(\dot{\theta}_1 + \dot{\theta}_1) \\ 0 \end{bmatrix} $$

$$ {}^{3}\upsilon_3 = {}^{3}_{2}R({}^{2}\upsilon_2 + {}^{2}\omega_2 \times {}^{2}P_3) = \begin{bmatrix} l_1 s_2 \dot{\theta}_1 \\ l_1 c_2 \dot{\theta}_1 + l_2(\dot{\theta}_1 + \dot{\theta}_1) \\ 0 \end{bmatrix} \tag{5.55}$$

$$ {}^{0}_{3}R = \begin{bmatrix} c_{12} & -s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.56}$$

$$ {}^{0}\upsilon_3 = {}^{3}_{2}R({}^{2}\upsilon_2 + {}^{2}\omega_2 \times {}^{2}P_3) = \begin{bmatrix} -l_1 s_1 \dot{\theta}_1 - l_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_1) \\ l_1 c_1 \dot{\theta}_1 + l_2 c_{12}(\dot{\theta}_1 + \dot{\theta}_1) \\ 0 \end{bmatrix} \tag{5.57}$$

Note that the components of ${}^{0}\upsilon_3$ may also be determined geometrically.

Jacobians

Given a system of six non-linear equations for six X's and six Y's.

$$ Y = F(X) $$

$$ \delta Y = \frac{\delta F}{\delta X}\delta X $$

$$ \delta Y = J(X)\delta X \qquad \dot{Y} = J(X)\dot{X} $$

A Jacobian in robotics is a matrix of partial derivatives that maps a joint velocity vector

$$ {}^{0}\dot{\Theta} = \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 & \cdots & \dot{\theta}_n \end{bmatrix}^{T} $$

into a velocity vector of the end-effector:

$$^{0}\upsilon = \begin{bmatrix} \upsilon_x & \upsilon_y & \upsilon_z & \omega_x & \omega_y & \omega_z \end{bmatrix}^{T}$$

$$^{0}\upsilon = {}^{0}J(\Theta)\dot{\Theta}$$

$$^{0}\dot{\Theta} = {}^{0}J(\Theta)^{-1}\,{}^{0}\upsilon$$

$$^{0}J(\Theta) = \begin{bmatrix} \frac{\partial f_1}{\partial \theta_1} & \frac{\partial f_1}{\partial \theta_2} & \cdots\cdots & \frac{\partial f_1}{\partial \theta_6} \\ \frac{\partial f_2}{\partial \theta_1} & \frac{\partial f_2}{\partial \theta_2} & \cdots\cdots & \frac{\partial f_2}{\partial \theta_6} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{\partial f_6}{\partial \theta_1} & \frac{\partial f_6}{\partial \theta_2} & \cdots\cdots & \frac{\partial f_6}{\partial \theta_6} \end{bmatrix} \text{ for six axis robots}$$

Changing a Jacobian reference frame in {B} to frame {A}:

$$\begin{bmatrix} {}^{A}\upsilon \\ {}^{A}\omega \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R & 0 \\ 0 & {}^{A}_{B}R \end{bmatrix} \begin{bmatrix} {}^{B}\upsilon \\ {}^{B}\omega \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R & 0 \\ 0 & {}^{A}_{B}R \end{bmatrix} {}^{B}J(\Theta)\dot{\Theta} \qquad (5.70)$$

$$\text{Thus, } {}^{A}J(\Theta) = \begin{bmatrix} {}^{A}_{B}R & 0 \\ 0 & {}^{A}_{B}R \end{bmatrix} {}^{B}J(\Theta) \qquad (5.71)$$

Find $^{3}J(\Theta) = \begin{bmatrix} \quad \end{bmatrix}$ and $^{0}J(\Theta) = \begin{bmatrix} \quad \end{bmatrix}$ from (5.55) and (5.57)

Singularities

$$\dot{\Theta} = J(\Theta)^{-1}\upsilon \qquad (5.72)$$

If a matrix is singular, its determinant is zero, and so its inverse cannot be found. As such, under a singular condition, the end effector velocity cannot be translated into the joint angular velocity.

Types of singularities:

1) Work space boundary singularity – Occurs when the robot arm is fully stretched out with the end effector reaching the outer boundary.
2) Work space interior singularity – Occurs when two links line up to fold with the end effector inside the work space boundary.

Static Forces in Manipulators

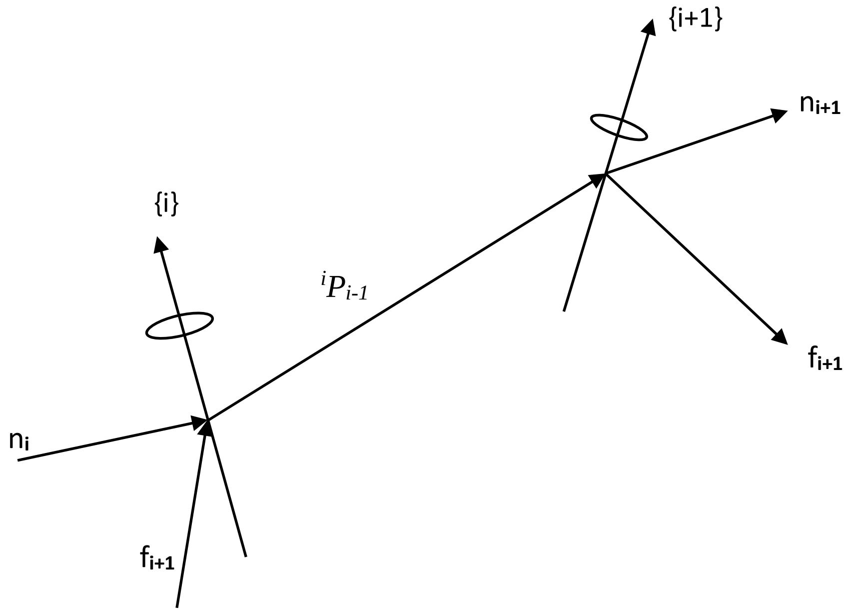
$$n = P \times f$$
$$n = |P||f|\sin\theta$$ ------ a cross vector product

${}^{i}n_i$ = torque exerted on link i by link i-1, express in {i}

${}^{i}f_i$ = force exerted on link i by link i-1, express in {i}

θ_i = angle between ${}^{i}f_i$ and ${}^{i}\eta_i$

${}^{i}P_{i+1}$ = displacement of link i+1, viewed in {i}



Equilibrium (counter balancing) of Force & Moment at a single link - Propagation Equations:

$${}^{i}f_i = {}^{i}f_{i+1} = {}_{i+1}^{i}R\,{}^{i+1}f_{i+1} \quad (5.80)$$

$${}^{i}n_i = {}^{i}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_{i+1} = {}_{i+1}^{i}R\,{}^{i+1}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_{i+1} \quad (5.81)$$

Joint torque at equilibrium –revolute joints = (Moment vector) (Joint axis vector)

$$\tau_i = {}^{i}n_i^{T}\,{}^{i}\hat{z}_i \quad (5.82)$$

Joint torque at equilibrium –prismatic joints = (Force vector) (Joint axis vector)

$$\tau_i = {}^{i}f_i^{T}\,{}^{i}\hat{z}_i \quad (5.83)$$

The partial derivatives of (5.82) constitute a Jacobian $J(\Theta)$ *asin Ex. 5.7.*

Development of Jacobian for converting Force into Torque

Work done in Cartesian space = Work done in Joint space

$$F \cdot \delta X = \tau \cdot \delta\Theta \qquad (\textit{6 x 1 vectors}) \tag{5.91}$$

Rewriting in notation for matrix multiplication

$$F^T \delta X = \tau^T \delta\Theta \tag{5.94}$$

Since $\delta X = J\delta\theta$ by definition,

$$F^T J \delta\theta = \tau^T \delta\Theta \tag{5.95}$$

Since $\delta\theta \equiv \delta\Theta$,

$$F^T J = \tau^T$$

Transposing the two sides, $(F^T J)^T = (\tau^T)^T$

$$\tau = J^T F \tag{5.96}$$

A Jacobian transpose maps the gripper force into equivalent joint torques.

Force and Velocity Transformation in the tool frame

{A}=Revolute, {B}=Fixed, per Fig. 5.13

(6x1) velocity vector: $\nu = \begin{bmatrix} \upsilon \\ \omega \end{bmatrix}$, and (6x1) force/moment vector: $\bar{F} = \begin{bmatrix} F \\ N \end{bmatrix}$

For the **Velocity Transformation**, starting from (5.45) and (5.47)

$$^B\omega_B = {}^B_AR\,{}^A\omega_A + \dot{\theta}_B\,{}^B\hat{Z}_B \tag{5.45}$$

$$^B\upsilon_B = {}^B_AR({}^A\upsilon_B + {}^A\omega_A \times {}^AP_B) \tag{5.47}$$

Setting $\dot{\theta}_B = 0$.......(why?) and

$$\begin{bmatrix} ^B\upsilon_B \\ ^B\omega_B \end{bmatrix} = \begin{bmatrix} ^B_AR & -{}^B_AR\,{}^AP_{BOrg}\times \\ 0 & ^B_AR \end{bmatrix} \begin{bmatrix} ^A\upsilon_A \\ ^A\omega_A \end{bmatrix} \tag{5.101}$$

$$= \begin{bmatrix} ^B_AR\,{}^A\upsilon_A - {}^B_AR({}^AP_{BOrg} \times {}^A\omega_A) \\ ^B_AR\,{}^A\omega_A \end{bmatrix}$$

Note that

a) $^B_AR({}^A\omega_A \times {}^AP_B) = {}^B_AR(-{}^AP_B \times {}^A\omega_A) = -{}^B_AR\,{}^AP_B \times {}^A\omega_A$

b) $^AP_{BOrg} \times {}^A\omega_A = \begin{bmatrix} i & j & k \\ P_x & P_y & P_x \\ \Omega_x & \Omega_y & \Omega_z \end{bmatrix} = \begin{bmatrix} i(P_y\Omega_z + P_z\Omega_y) \\ j(P_z\Omega_x - P_x\Omega_z) \\ k(P_x\Omega_y - P_y\Omega_x) \end{bmatrix} = \begin{bmatrix} 0 & -P_z & P_y \\ P_z & 0 & -P_x \\ -P_y & P_x & 0 \end{bmatrix} \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix}$

$$\begin{bmatrix} ^A\upsilon_A \\ ^A\omega_A \end{bmatrix} = \begin{bmatrix} ^A_BR\,{}^B\upsilon_B + {}^A_BR({}^B_AR\,{}^AP_{BOrg} \times {}^A\omega_A) \\ ^A_BR\,{}^B\omega_B \end{bmatrix} = \begin{bmatrix} ^A_BR\,{}^B\upsilon_B + {}^AP_{BOrg} \times {}^A_BR\,{}^B\omega_B) \\ ^A_BR\,{}^B\omega_B \end{bmatrix}$$

$$= \begin{bmatrix} {}^A_B R & {}^A P_{BOrg} \times {}^A_B R \\ 0 & {}^A_B R \end{bmatrix} \begin{bmatrix} {}^B \upsilon_B \\ {}^B \omega_B \end{bmatrix} \qquad (5.102)$$

The **Force-Moment transformation** is derived from (5.80) and (5.81):

$${}^i f_i = {}^i f_{i+1} = {}^i_{i+1} R \, {}^{i+1} f_{i+1} \qquad (5.80)$$

$${}^i n_i = {}^i n_{i+1} + {}^i P_{i+1} \times {}^i f_{i+1} = {}^i_{i+1} R \, {}^{i+1} n_{i+1} + {}^i P_{i+1} \times {}^i f_{i+1} \qquad (5.81)$$

$$\begin{bmatrix} {}^A F_A \\ {}^A N_A \end{bmatrix} = \begin{bmatrix} {}^A_B R & 0 \\ {}^A P_{BOrg} \times {}^A_B R & {}^A_B R \end{bmatrix} \begin{bmatrix} {}^B F_B \\ {}^B N_B \end{bmatrix} \qquad (5.105)$$

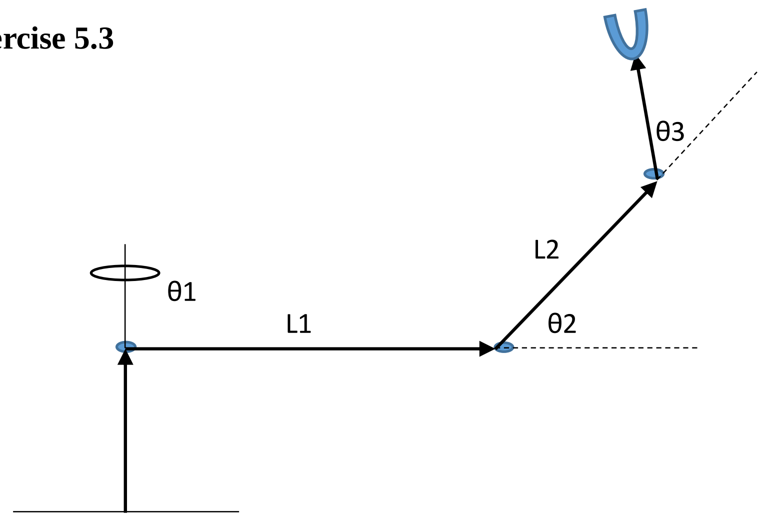
The relationship between Velocity transformation and Force-Moment transformation:

$${}^A_B T_f = ({}^A_B T_\upsilon)^T \qquad (5.107)$$

Homework #7 due 10/22/14

5.3(Jacobian from velocity propagation only), 5.13 (study only, the answer in the textbook), 5.15 (variables are θ'_1 and d'_2), 5.18 (only the 4th column is pertinent), 5.19 (same as 5.15)

Exercise 5.3



Jacobian derived from the velocity propagation from Base to Tip

A Homework #7 problem – Follow the procedure in Example 5.3.

Jacobian derived from Static Force propagation from Tip to Base

$${}^4 F_4 = [F_x \quad F_y \quad F_z]^T \qquad {}^4 N_4 = [0 \quad 0 \quad 0]$$

$${}^{3}F_{3} = {}^{3}_{4}R\,{}^{4}F_{4} = \begin{bmatrix} F_x & F_y & F_z \end{bmatrix}^T$$

$${}^{3}N_{3} = {}^{3}_{4}R\,{}^{4}N_{4} + {}^{3}_{4}P \times {}^{3}F_{3} = 0 + \begin{bmatrix} L_3 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} 0 \\ -L_3F_z \\ -L_3F_y \end{bmatrix}$$

$${}^{2}F_{2} = {}^{2}_{3}R\,{}^{3}F_{3} = \begin{bmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} c_3F_x - s_3F_y \\ s_3F_x + c_3F_y \\ F_z \end{bmatrix}$$

$${}^{2}N_{2} = {}^{2}_{3}R\,{}^{3}N_{3} + {}^{2}_{3}P \times {}^{2}F_{2} = {}^{2}_{3}R \begin{bmatrix} 0 \\ -L_3F_z \\ -L_3F_{z\backslash y} \end{bmatrix} + \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} c_3F_x - s_3F_y \\ s_3F_x + c_3F_y \\ F_z \end{bmatrix} = \begin{bmatrix} L_3s_3F_z \\ -L_3F_z - L_3c_3F_z \\ L_2(s_3F_x + c_3F_y) + L_3F_y \end{bmatrix}$$

$${}^{1}F_{1} = {}^{1}_{2}R\,{}^{2}F_{2} = \begin{bmatrix} c_2 & -s_2 & 0 \\ 0 & 0 & -1 \\ s_2 & c_2 & 0 \end{bmatrix} \begin{bmatrix} c_3F_x - s_3F_y \\ s_3F_x + c_3F_y \\ F_z \end{bmatrix} = \begin{bmatrix} c_2(c_3F_x - s_3F_y) - s_2(s_3F_x + c_3F_y) \\ F_z \\ s_2(c_3F_x - s_3F_y) + c_2(s_3F_x + c_3F_y) \end{bmatrix}$$

$${}^{1}N_{1} = {}^{1}_{2}R\,{}^{2}N_{2} + {}^{1}_{2}P \times {}^{1}F_{1} = {}^{1}_{2}R \begin{bmatrix} 0 \\ -L_3F_z \\ -L_3F_{z\backslash y} \end{bmatrix} + \begin{bmatrix} L_3s_3F_z \\ -L_3F_z - L_3c_3F_z \\ L_2(s_3F_x + c_3F_y) + L_3F_y \end{bmatrix} + \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} \times {}^{1}F_{1} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

Torques τ_1, τ_2, τ_3 = the Z elements of ${}^{1}N_{1}$, ${}^{2}N_{2}$, and ${}^{3}N_{3}$

$$\tau_1 = [-L_1 - L_2c_2 + L_3(s_2s_3 - c_2c_3)]F_z = (-L_1 - L_2c_2 - L_3c_{23})F_z$$

$$\tau_2 = L_2s_3F_x + (L_2c_3 + L_3)F_y$$

$$\tau_3 = L_3F_y$$

Rearranging, $\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -L_1 - L_2c_2 - L_3c_{23} \\ L_2s_3 & L_2c_3 + L_3 & 0 \\ 0 & L_3 & 0 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = {}^{4}J^T(\theta) \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} g,$

Since $\tau = J^T F$

$\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} = {}^{4}J^T(\theta) \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}$, therefore a transpose of the (3x3) Jacobian matrix is now found.

Jacobian derived from direct Differentiation of the kinematic equations

By observation of the geometric link-frame diagram, the kinematric equations are:

$$ {}^0P_{4Org} = \begin{bmatrix} {}^4P_x \\ {}^4P_y \\ {}^4P_z \end{bmatrix} = \begin{bmatrix} L_1c_1 + L_2c_1c_2 + L_3c_1c_{23} \\ L_1s_1 + L_2s_1c_2 + L_3s_1c_{23} \\ L_2s_2 + L_3s_{23} \end{bmatrix} $$

Taking partial derivatives to arrive at a Jacobian,

$$ {}^0J(\theta) = \begin{bmatrix} \frac{\partial P_x}{\partial \theta_1} & \frac{\partial P_x}{\partial \theta_2} & \frac{\partial P_x}{\partial \theta_3} \\ \frac{\partial P_y}{\partial \theta_1} & \frac{\partial P_y}{\partial \theta_2} & \frac{\partial P_y}{\partial \theta_3} \\ \frac{\partial P_z}{\partial \theta_1} & \frac{\partial P_z}{\partial \theta_2} & \frac{\partial P_z}{\partial \theta_3} \end{bmatrix} $$

Once ${}^0J(\theta)$ is found, ${}^4J(\theta)$ can be found from:

${}^4J(\theta) = {}^4_0R\,{}^0J(\theta)$,

where ${}^4_0R = {}^0_4R^T$ is readily calculated from the rotational matrixes.