Beam Deflections Using Double Integration

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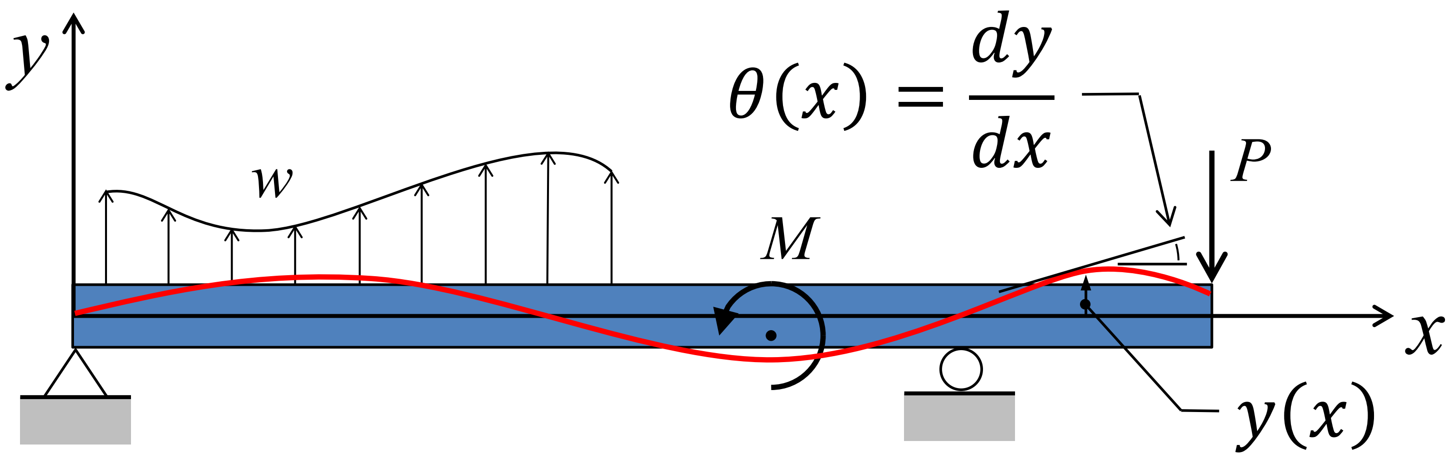
Recall the Moment-Curvature Relationship for Small Deformations

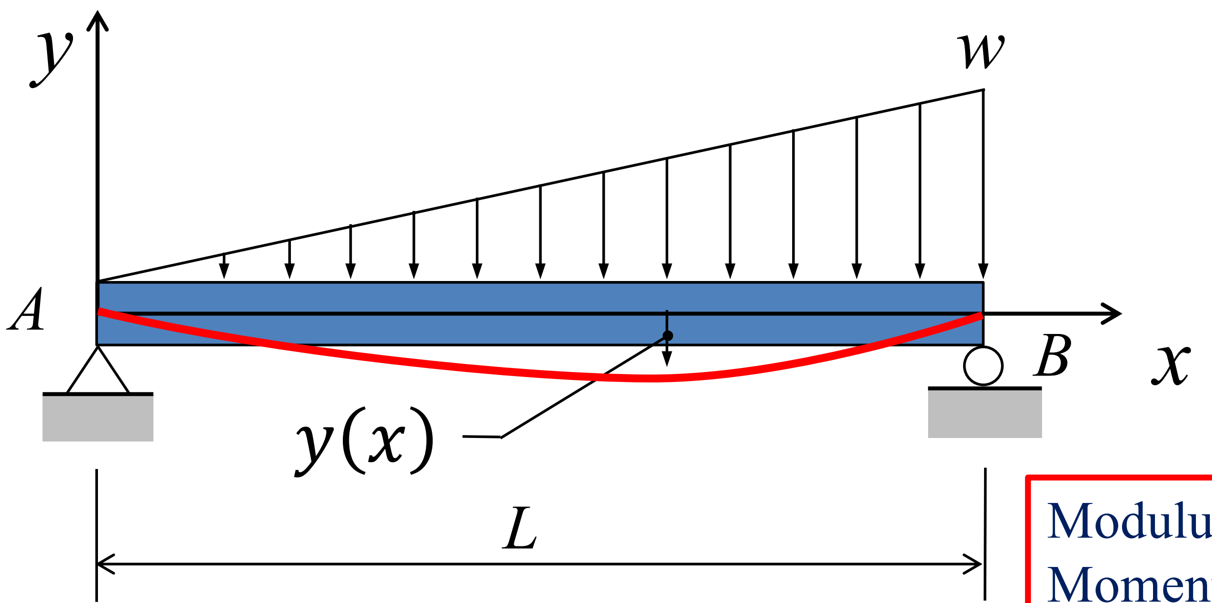


$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

In order to solve this differential equation for y and θ we need:

- Moment equation (from statics);
- Two boundary (or continuity) conditions on y or θ;
- Information on E and I.

Example Problem

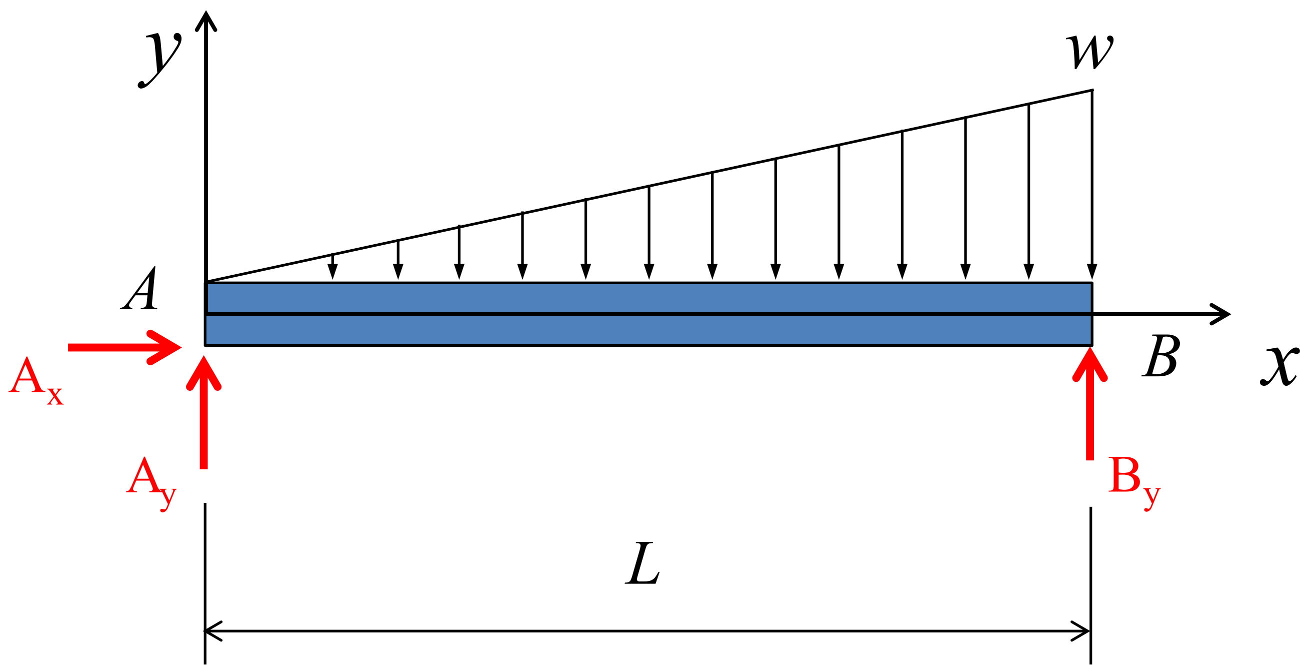


Modulus of Elasticity = E
Moment of Inertia = I

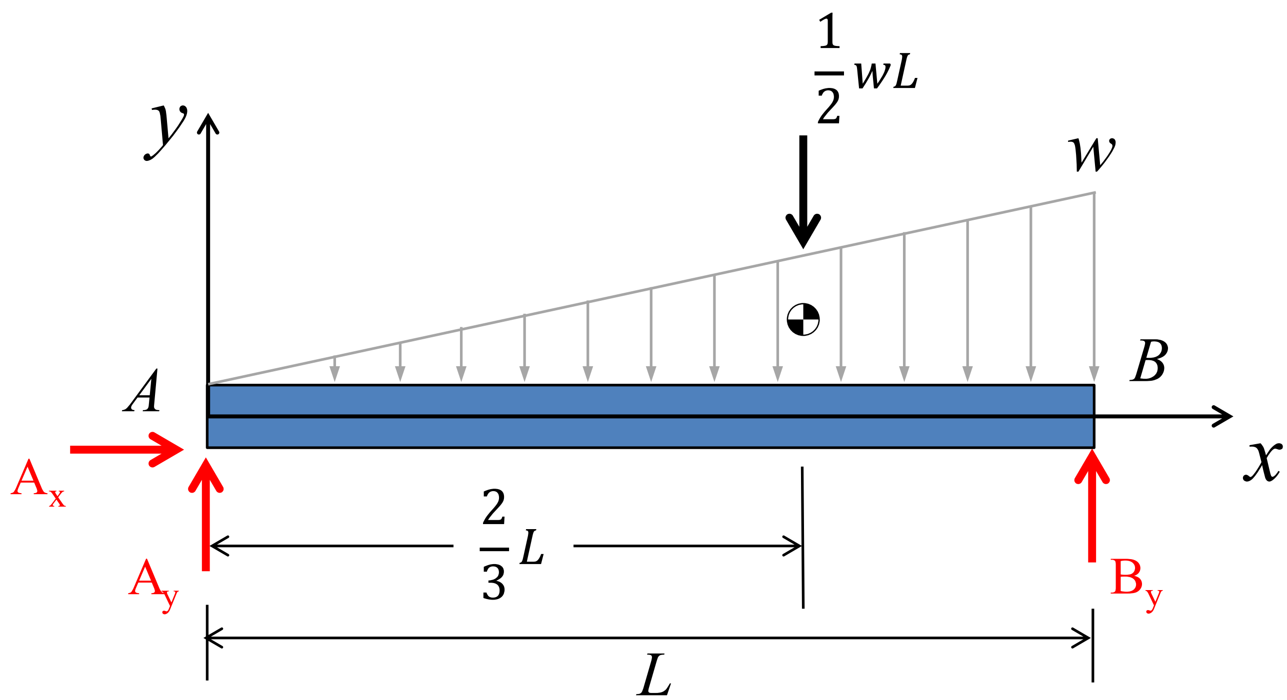
Find the equation of the elastic curve for the simply supported beam subjected to the uniformly distributed load using the double integration method. Find the maximum deflection. EI is constant.

Free Body Diagram of the Beam

Need to find the moment function $M(x)$



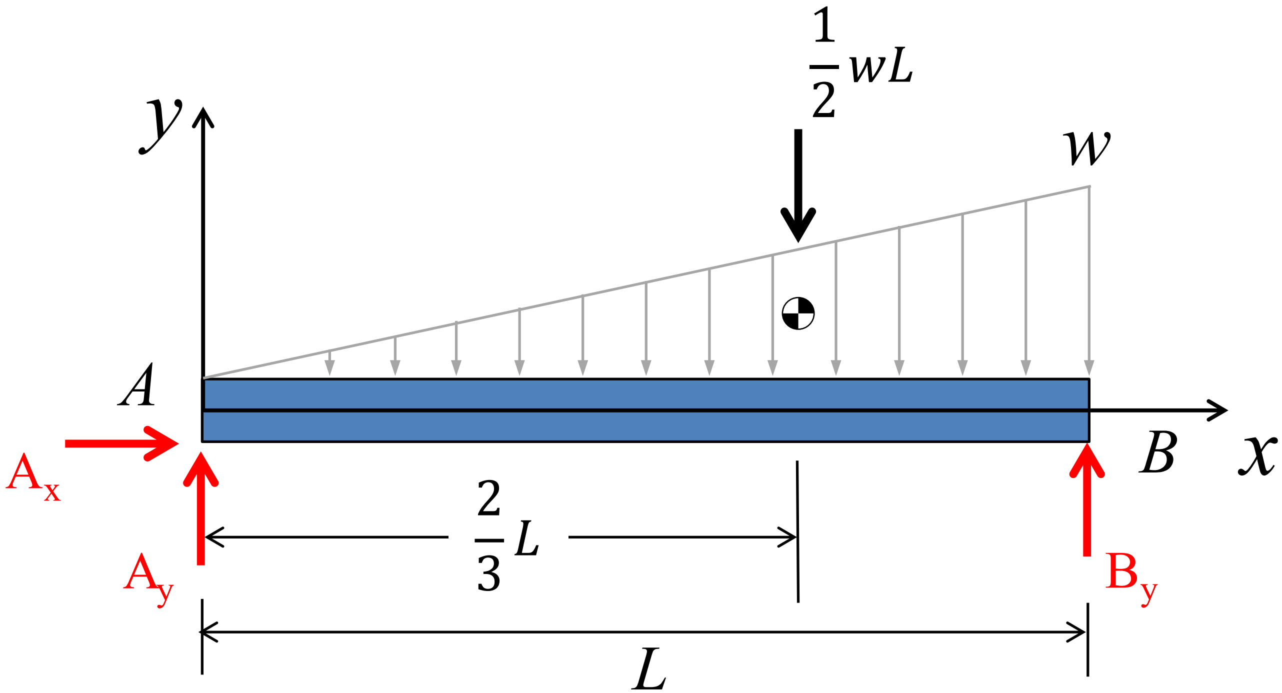
Free Body Diagram of the Beam



$$\circlearrowleft + \sum M_A = 0$$

$$B_y = \frac{1}{3}wL$$

Free Body Diagram of the Beam

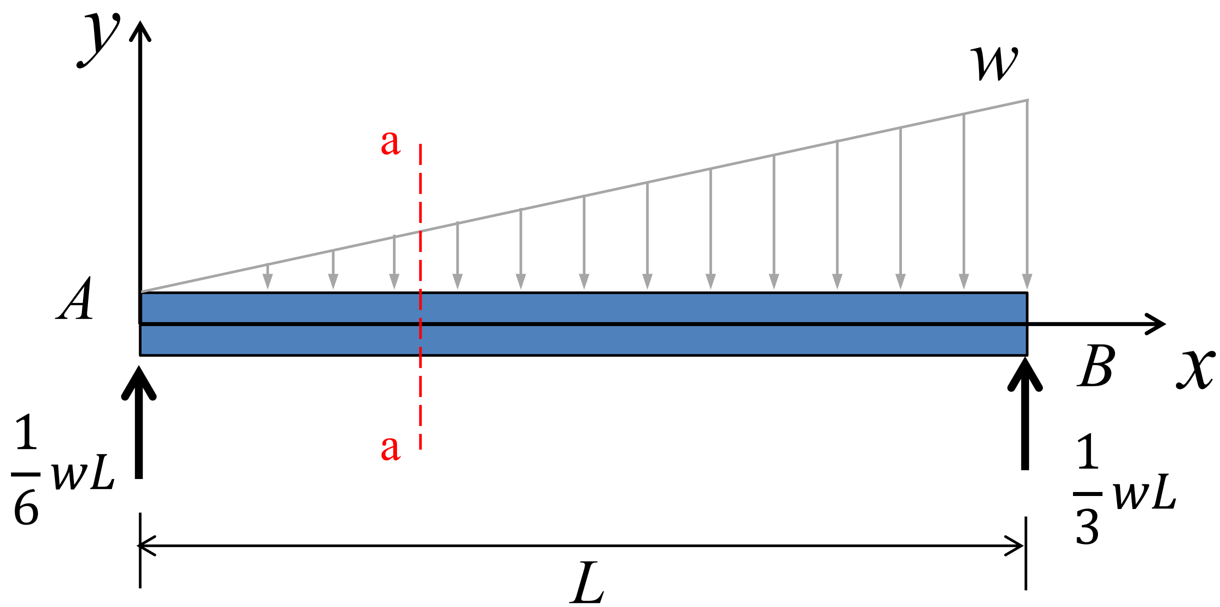


$$+\uparrow \sum F_y = 0$$

$$\xrightarrow{+} \sum F_x = 0$$

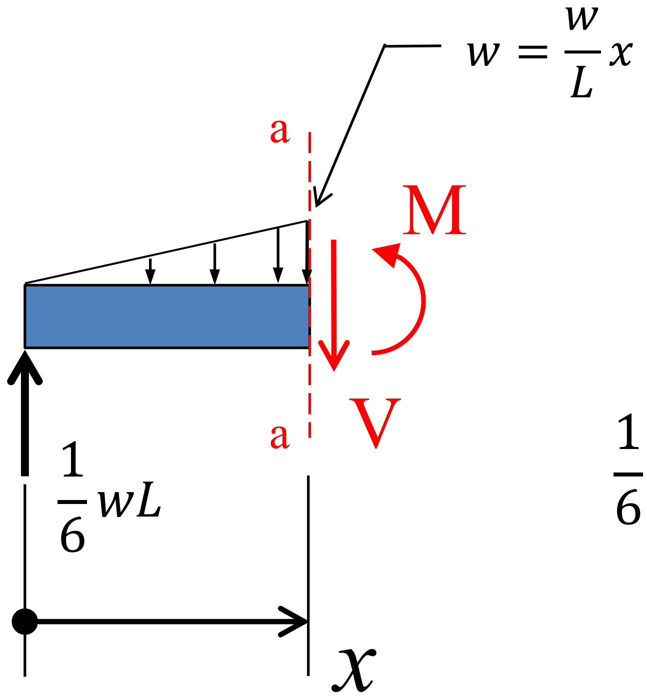
$$A_y = \frac{1}{6}wL$$

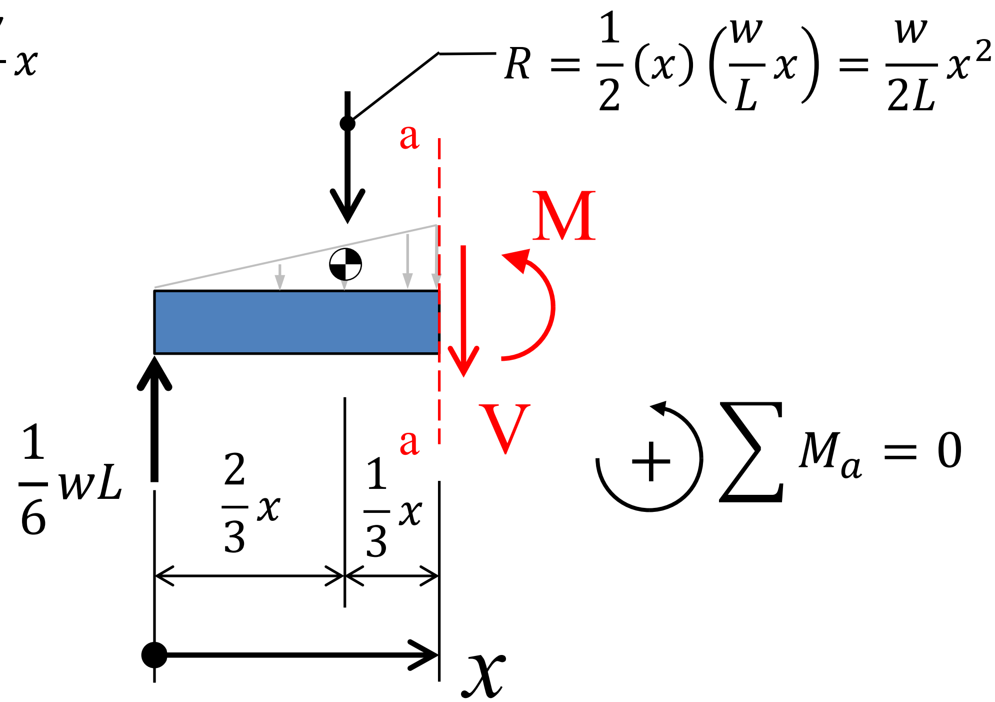
Free Body Diagram of the Beam Showing Known Reactions



Cut the beam at *a*-*a* to find the moment function $M(x)$

Free Body Diagram of Segment to the Left of *a–a*

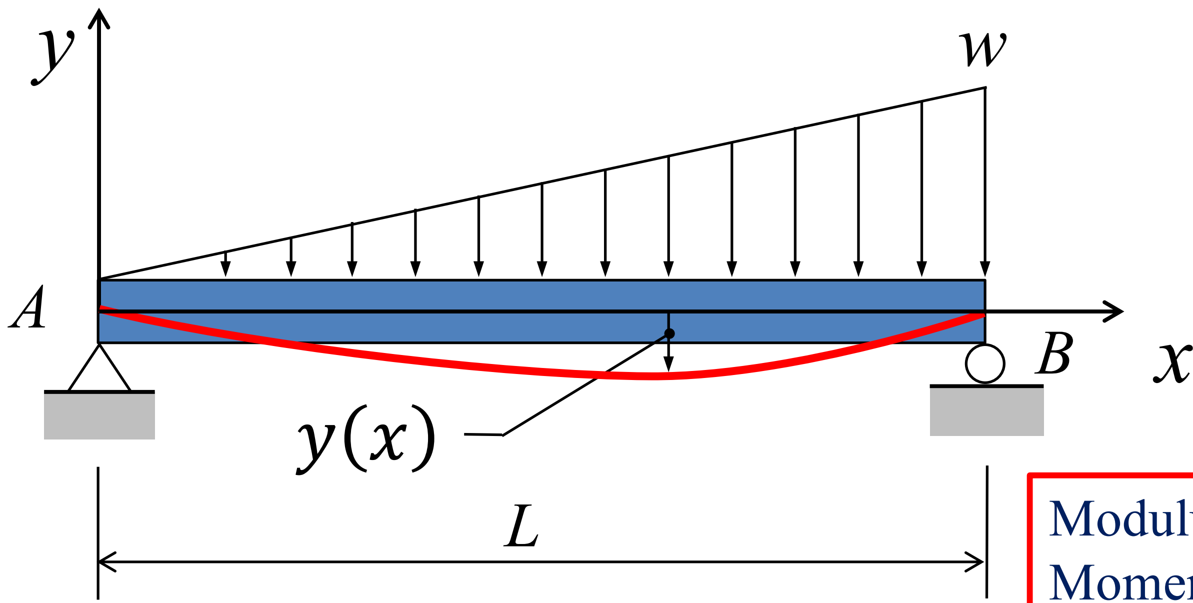




$$\circlearrowleft + \sum M_a = 0$$

$$M = \frac{wL}{6}x - \frac{w}{6L}x^3$$

Need Two Boundary Conditions on y or θ

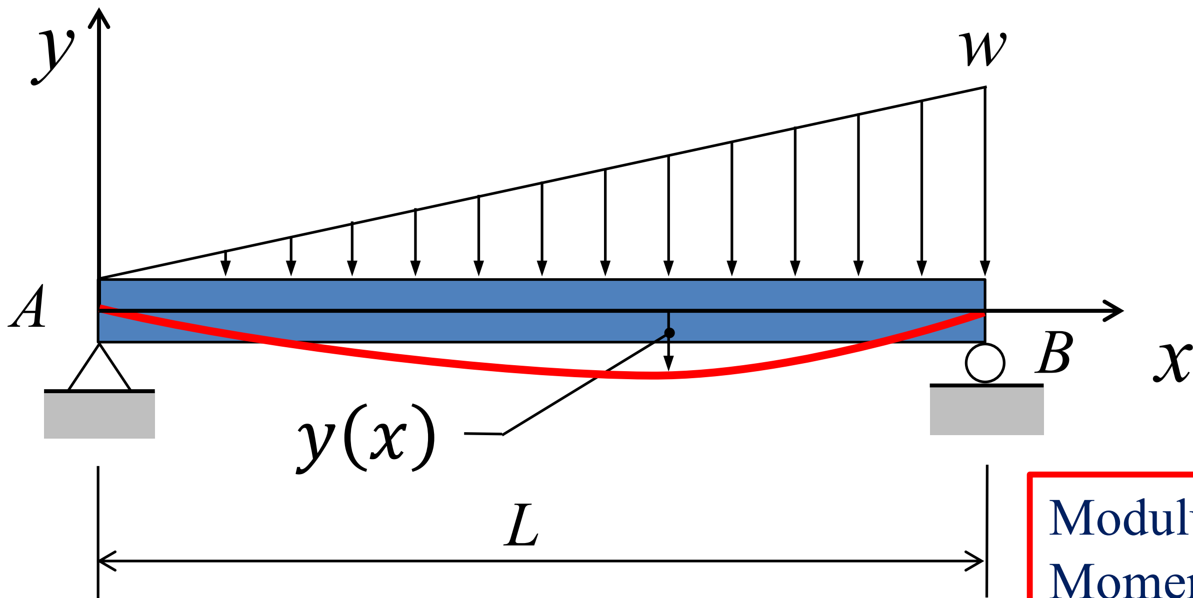


Modulus of Elasticity = E
Moment of Inertia = I

$$y(0) = 0$$

$$y(L) = 0$$

Solve the Differential Equation



Modulus of Elasticity = E
Moment of Inertia = I

For constant EI

$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

$$EI\frac{d^2y}{dx^2} = M$$

$$y(L) = 0$$

$$y(0) = 0$$

$$EI\frac{d^2y}{dx^2} = \frac{wL}{6}x - \frac{w}{6L}x^3$$

$$M = \frac{wL}{6}x - \frac{w}{6L}x^3$$

First Integration

$$EI\frac{d^2y}{dx^2} = \frac{wL}{6}x - \frac{w}{6L}x^3$$

$$\int EI\frac{d^2y}{dx^2}dx = \int \left(\frac{wL}{6}x - \frac{w}{6L}x^3\right)dx$$

$$EI\frac{dy}{dx} = -\frac{w}{24L}x^4 + \frac{wL}{12}x^2 + C_1$$

$$EI\theta = -\frac{w}{24L}x^4 + \frac{wL}{12}x^2 + C_1$$

Second Integration

$$EI\frac{dy}{dx} = -\frac{w}{24L}x^4 + \frac{wL}{12}x^2 + C_1$$

$$\int EI\frac{dy}{dx}dx = \int \left(-\frac{w}{24L}x^4 + \frac{wL}{12}x^2 + C_1\right)dx$$

$$EIy = -\frac{w}{120L}x^5 + \frac{wL}{36}x^3 + C_1x + C_2$$

Use the two boundary conditions to find C_1 *and* C_2

$$y(0) = 0$$

$$y(L) = 0$$

Find Constants C_1 and C_2

$$EIy = -\frac{w}{120L}x^5 + \frac{wL}{36}x^3 + C_1 x + C_2$$

$$y(0) = 0$$

$$EI(0) = -\frac{w}{120L}(0)^5 + \frac{wL}{36}(0)^3 + C_1(0) + C_2$$

$$EI(0) = -\frac{w}{120L}(0)^5 + \frac{wL}{36}(0)^3 + C_1(0) + C_2$$

$$C_2 = 0$$

$$EI(0) = -\frac{w}{120L}(L)^5 + \frac{wL}{36}(L)^3 + C_1(L) + C_2$$

$$y(L) = 0$$

$$C_1 = \frac{wL^3}{120} - \frac{wL^3}{36} = -\frac{7wL^3}{360}$$

Functions for y and θ

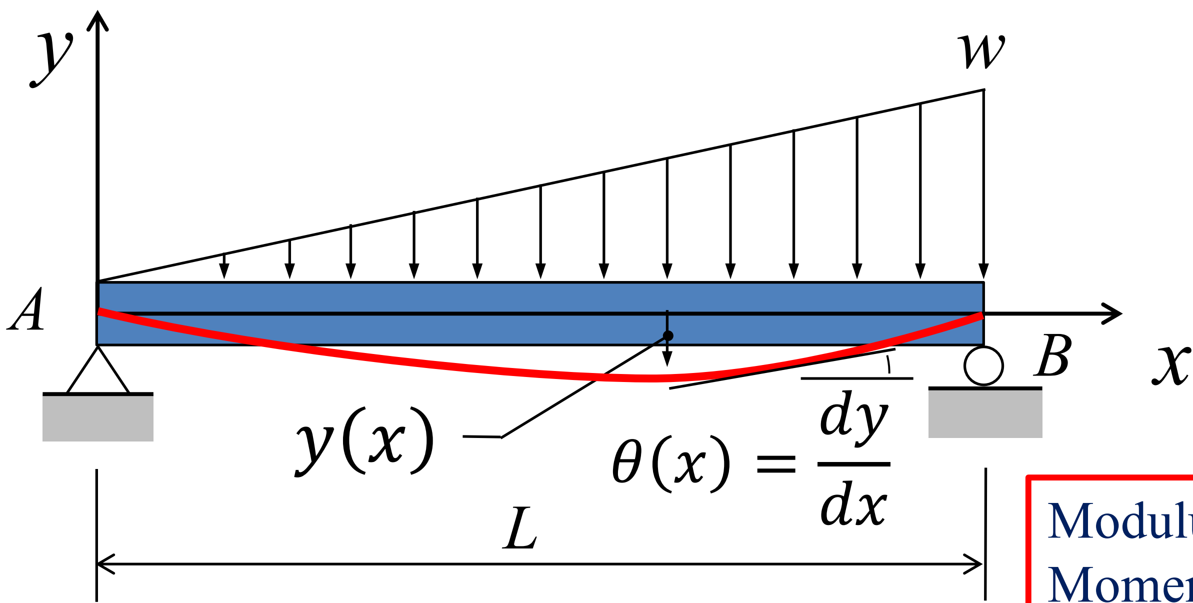
$$EIy = -\frac{w}{120L}x^5 + \frac{wL}{36}x^3 - \frac{7wL^3}{360}x$$

$$EI\theta = -\frac{w}{24L}x^4 + \frac{wL}{12}x^2 - \frac{7wL^3}{360}$$

$$y(x) = \frac{1}{EI}\left(-\frac{w}{120L}x^5 + \frac{wL}{36}x^3 - \frac{7wL^3}{360}x\right)$$

$$\theta(x) = \frac{1}{EI}\left(-\frac{w}{24L}x^4 + \frac{wL}{12}x^2 - \frac{7wL^3}{360}\right)$$

Functions for y and θ

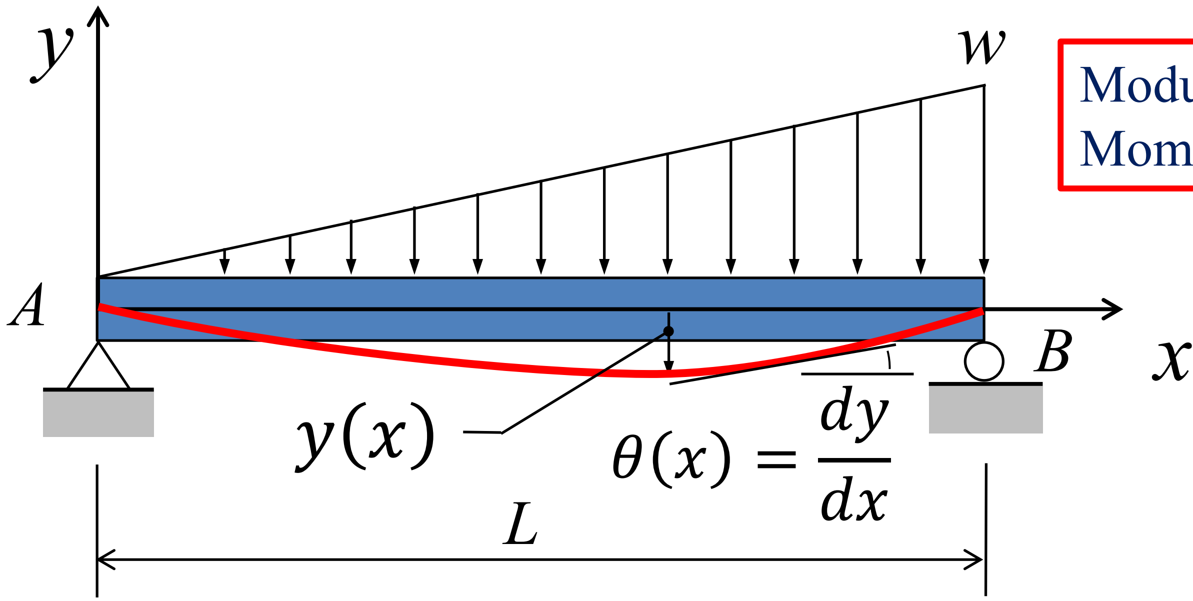


Modulus of Elasticity = E
Moment of Inertia = I

$$y(x) = \frac{1}{EI}\left(-\frac{w}{120L}x^5 + \frac{wL}{36}x^3 - \frac{7wL^3}{360}x\right)$$

$$\theta(x) = \frac{1}{EI}\left(-\frac{w}{24L}x^4 + \frac{wL}{12}x^2 - \frac{7wL^3}{360}\right)$$

Questions



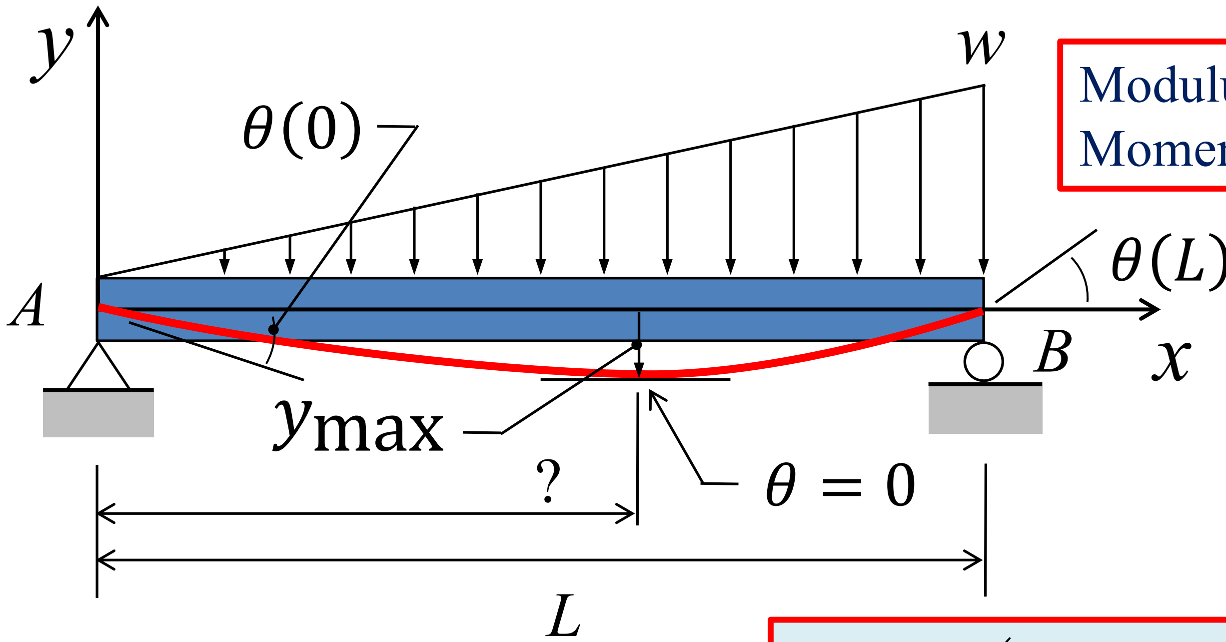
Modulus of Elasticity = E
Moment of Inertia = I

$$y(x) = \frac{1}{EI}\left(-\frac{w}{120L}x^5 + \frac{wL}{36}x^3 - \frac{7wL^3}{360}x\right)$$

$$\theta(x) = \frac{1}{EI}\left(-\frac{w}{24L}x^4 + \frac{wL}{12}x^2 - \frac{7wL^3}{360}\right)$$

How can we find the maximum displacement?
Where does the maximum displacement occur?
How can we find the rotation at the supports?

Answers



Modulus of Elasticity = E
Moment of Inertia = I

$$y(x) = \frac{1}{EI}\left(-\frac{w}{120L}x^5 + \frac{wL}{36}x^3 - \frac{7wL^3}{360}x\right)$$

$$\theta(x) = \frac{1}{EI}\left(-\frac{w}{24L}x^4 + \frac{wL}{12}x^2 - \frac{7wL^3}{360}\right)$$

y_{max} occurs where $\theta = 0$

θ_A is $\theta(0)$
θ_B is $\theta(L)$

Point Where Maximum Deflection Occurs

Set $\theta = 0$

$$\theta(x) = \frac{1}{EI}\left(-\frac{w}{24L}x^4 + \frac{wL}{12}x^2 - \frac{7wL^3}{360}\right) = \mathbf{0}$$

Let $q = x^2$

Multiply both sides by $\frac{EIL}{w}$

$$-\frac{1}{24}q^2 + \frac{L^2}{12}q - \frac{7L^4}{360} = \mathbf{0}$$

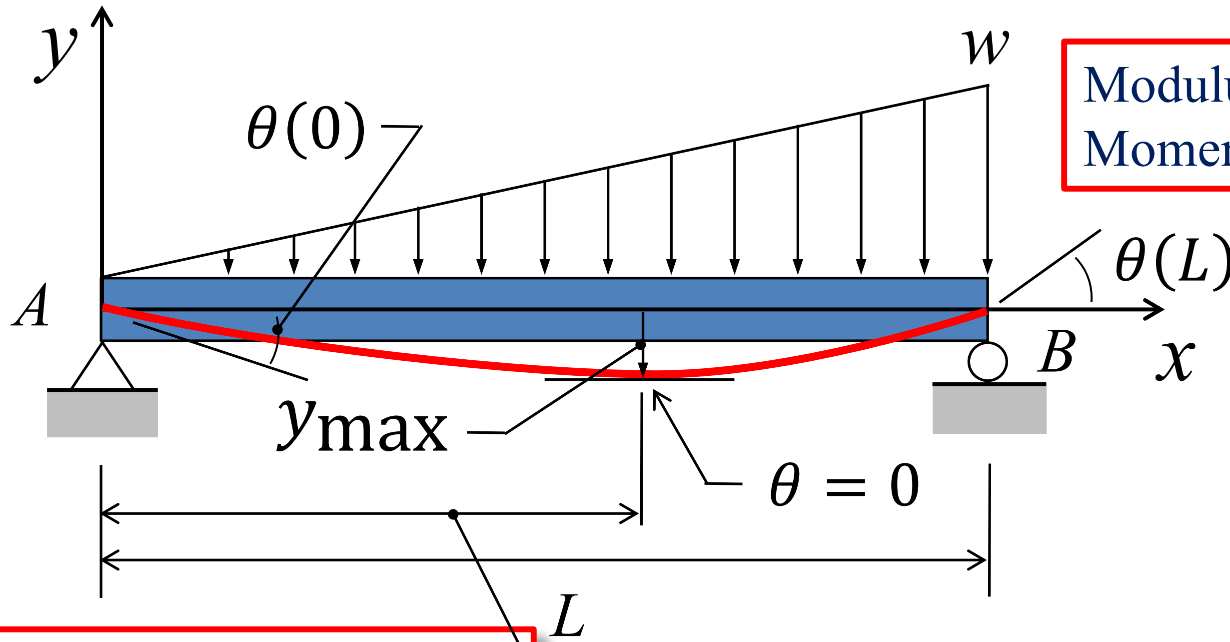
$$q = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$q = \frac{-\frac{L^2}{12} \pm \sqrt{\frac{L^4}{144} - 4\left(-\frac{1}{24}\right)\left(-\frac{7L^4}{360}\right)}}{2\left(-\frac{1}{24}\right)} = L^2 \pm 12\sqrt{\frac{L^4}{144} - \left(\frac{1}{6}\right)\left(\frac{7L^4}{360}\right)} = L^2 \pm 12L^2\sqrt{\frac{1}{270}}$$

$$q = L^2 \pm 4L^2\sqrt{\frac{1}{30}} = \left(1 + 4\frac{1}{\sqrt{30}}, \mathbf{1 - 4\frac{1}{\sqrt{30}}}\right)L^2$$

$$x = \left(\sqrt{1 - 4\frac{1}{\sqrt{30}}}\right)L = 0.51893L$$

Maximum Deflection

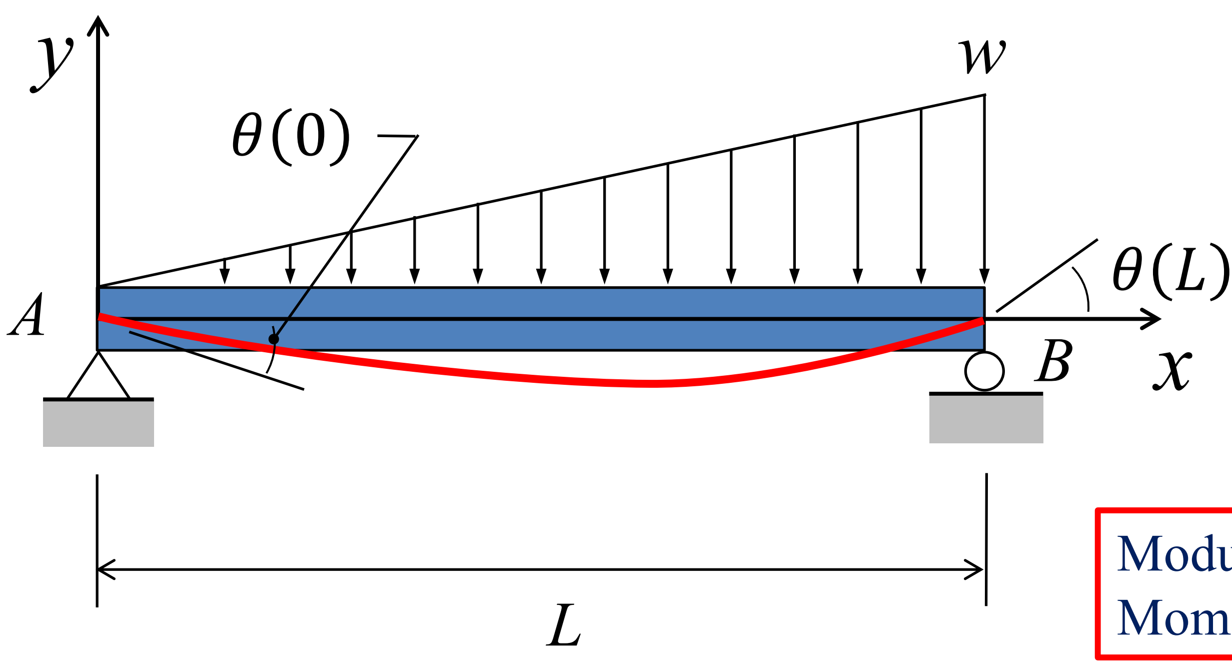


Modulus of Elasticity = E
Moment of Inertia = I

$$\left(\sqrt{1 - 4\frac{1}{\sqrt{30}}}\right)L = 0.51893L$$

$$y_{\max} = \frac{1}{EI}\left(-\frac{w}{120L}(0.5193L)^5 + \frac{wL}{36}(0.5193L)^3 - \frac{7wL^3}{360}(0.5193L)\right) = -0.006522\frac{wL^4}{EI}$$

Slope at Supports



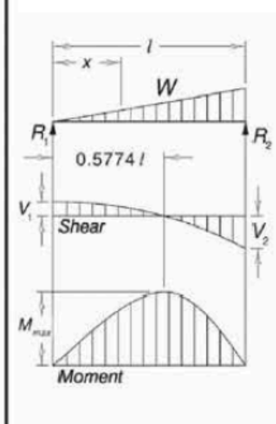
Modulus of Elasticity = E
Moment of Inertia = I

$$\theta_A = \theta(0) = \frac{1}{EI}\left(-\frac{7wL^3}{360}\right) = -\frac{7wL^3}{360EI} = -0.01944\frac{wL^3}{EI}$$

$$\theta_B = \theta(L) = \frac{1}{EI}\left(-\frac{w}{24L}L^4 + \frac{wL}{12}L^2 - \frac{7wL^3}{360}\right) = \frac{wL^3}{45EI} = 0.02222\frac{wL^3}{EI}$$

Compare to Tabulated Solution in AISC Manual

2. SIMPLE BEAM — LOAD INCREASING UNIFORMLY TO ONE END



Total Equiv. Uniform Load	$= \frac{16W}{9\sqrt{3}} = 1.03\,W$
$R_1 = V_1$	$= \frac{W}{3}$
$R_2 = V_2 = V_{max}$	$= \frac{2W}{3}$
V_x	$= \frac{W}{3} - \frac{Wx^2}{l^2}$
M_{max} $\left(\text{at } x = \frac{l}{\sqrt{3}} = 0.577\,l\right)$	$= \frac{2Wl}{9\sqrt{3}} = 0.128\,Wl$
M_x	$= \frac{Wx}{3l^2}(l^2 - x^2)$
Δ_{max} $\left(\text{at } x = l\sqrt{1-\sqrt{\frac{8}{15}}} = 0.519\,l\right)$	$= 0.0130\,\frac{Wl^3}{EI}$
Δ_x	$= \frac{Wx}{180\,EIl^2}\left(3x^4 - 10l^2x^2 + 7l^4\right)$

$$W = \frac{1}{2}wL$$