Review of Statistical Inference

Appendix C

Prepared by Vera Tabakova, East Carolina University

Appendix C: Review of Statistical Inference

Appendix C: Review of Statistical Inference

C.1 A Sample of Data

Table C.1 **Sample Hip Size Data**

14.96	14.76	15.97	15.71	17.77
17.34	17.89	17.19	13.53	17.81
16.40	18.36	16.87	17.89	16.90
19.33	17.59	15.26	17.31	19.26
17.69	16.64	13.90	13.71	16.03
17.50	20.23	16.40	17.92	15.86
15.84	16.98	20.40	14.91	16.56
18.69	16.23	15.94	20.00	16.71
18.63	14.21	19.08	19.22	20.23
18.55	20.33	19.40	16.48	15.54

C.1 A Sample of Data

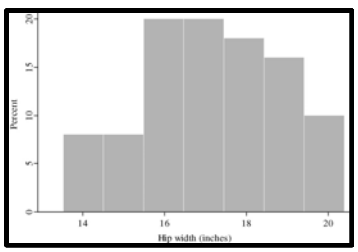

Figure C.1 Histogram of Hip Sizes

C.2 An Econometric Model

$$E[Y] = \mu \quad \text{(C.1)}$$

$$\operatorname{var}(Y) = E[Y - E(Y)]^2 = E[Y - \mu]^2 = \sigma^2 \quad \text{(C.2)}$$

C.3 Estimating the Mean of a Population

$$\bar{y} = \sum y_i / N \quad \text{(C.3)}$$

$$\bar{Y} = \sum_{i=1}^{N} Y_i / N \quad \text{(C.4)}$$

C.3 Estimating the Mean of a Population

$$\bar{y} = \sum y_i / N \quad \text{(C.3)}$$

$$\bar{Y} = \sum_{i=1}^{N} Y_i / N \quad \text{(C.4)}$$

C.3 Estimating the Mean of a Population

Table C.2 **Sample Means from 10 Samples**

Sample	$\bar{y}$
1	17.3544
2	16.8220
3	17.4114
4	17.1654
5	16.9004
6	16.9956
7	16.8368
8	16.7534
9	17.0974
10	16.8770

C.3.1 The Expected Value of $\bar{Y}$

$$\bar{Y} = \sum_{i=1}^{N} \frac{1}{N} Y_i = \frac{1}{N} Y_1 + \frac{1}{N} Y_2 + \ldots + \frac{1}{N} Y_N \qquad \text{(C.5)}$$

$$\begin{aligned} E[\bar{Y}] &= E\left[\frac{1}{N} Y_1\right] + E\left[\frac{1}{N} Y_2\right] + \ldots + E\left[\frac{1}{N} Y_N\right] \\ &= \frac{1}{N} E[Y_1] + \frac{1}{N} E[Y_2] + \ldots + \frac{1}{N} E[Y_N] \\ &= \frac{1}{N}\mu + \frac{1}{N}\mu + \ldots + \frac{1}{N}\mu \\ &= \mu \end{aligned}$$

C.3.2 The Variance of $\bar{Y}$

$$\begin{aligned} \operatorname{var}(\bar{Y}) &= \operatorname{var}\left(\frac{1}{N} Y_1 + \frac{1}{N} Y_2 + \ldots + \frac{1}{N} Y_N\right) \\ &= \frac{1}{N^2} \operatorname{var}(Y_1) + \frac{1}{N^2} \operatorname{var}(Y_2) + \ldots + \frac{1}{N^2} \operatorname{var}(Y_N) \\ &= \frac{1}{N^2}\sigma^2 + \frac{1}{N^2}\sigma^2 + \ldots + \frac{1}{N^2}\sigma^2 \\ &= \frac{\sigma^2}{N} \end{aligned} \qquad \text{(C.6)}$$

C.3.3 The Sampling Distribution of $\bar{Y}$

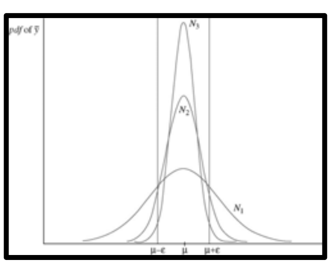

Figure C.2 Increasing Sample Size and Sampling Distribution of $\bar{Y}$

C.3.3 The Sampling Distribution of $\bar{Y}$

$$P\left[\mu-1 \le \bar{Y} \le \mu+1\right] = P\left[\frac{-1}{\sigma/\sqrt{N}} \le \frac{\bar{Y}-\mu}{\sigma/\sqrt{N}} \le \frac{1}{\sigma/\sqrt{N}}\right]$$
$$= P\left[\frac{-1}{\sqrt{.25}} \le Z \le \frac{1}{\sqrt{.25}}\right]$$
$$= P\left[-2 \le Z \le 2\right] = .9544$$

- If we draw a random sample of size $N = 40$ from a normal population with variance 10, the least squares estimator will provide an estimate within 1 inch of the true value about 95% of the time. If $N = 80$ the probability that $\bar{Y}$ is within 1 inch of μ increases to 0.995.

Principles of Econometrics, 3rd Edition *Slide C-13*

C.3.4 The Central Limit Theorem

Central Limit Theorem: If $Y_1,\ldots,Y_N$ are independent and identically distributed random variables with mean μ and variance σ^2, and $\bar{Y} = \sum Y_i / N$, then $Z_N = \dfrac{\bar{Y}-\mu}{\sigma/\sqrt{N}}$ has a probability distribution that converges to the standard normal $N(0,1)$ as $N \to \infty$.

Principles of Econometrics, 3rd Edition *Slide C-14*

C.3.4 The Central Limit Theorem

$$f(y) = \begin{cases} 2y & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$Z_N = \frac{\bar{Y} - 2/3}{\sqrt{\dfrac{1/18}{N}}}$$

Principles of Econometrics, 3rd Edition *Slide C-15*

C.3.4 The Central Limit Theorem

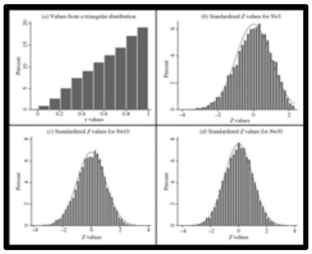

Figure C.3 Central Limit Theorem

C.3.5 Best Linear Unbiased Estimation

- A powerful finding about the estimator of the population mean is that it is the best of all possible estimators that are both *linear* and *unbiased*.
- A **linear estimator** is simply one that is a weighted average of the Y_i's, such as $\tilde{Y} = \sum a_i Y_i$, where the a_i are constants.
- "Best" means that it is the linear unbiased estimator with the smallest possible variance.

C.4 Estimating the Population Variance and Other Moments

$$\mu_r = E\left[(Y-\mu)^r\right]$$

$$\mu_1 = E\left[(Y-\mu)^1\right] = E(Y) - \mu = 0$$

$$\mu_2 = E\left[(Y-\mu)^2\right] = \sigma^2$$

$$\mu_3 = E\left[(Y-\mu)^3\right]$$

$$\mu_4 = E\left[(Y-\mu)^4\right]$$

C.4.1 Estimating the population variance

$$\operatorname{var}(Y) = \sigma^2 = E[Y-\mu]^2$$

$$\tilde{\sigma}^2 = \frac{\sum\left(Y_i - \bar{Y}\right)^2}{N}$$

$$\hat{\sigma}^2 = \frac{\sum\left(Y_i - \bar{Y}\right)^2}{N-1} \qquad \text{(C.7)}$$

Principles of Econometrics, 3rd Edition *Slide C-19*

C.4.1 Estimating the population variance

$$\widehat{\operatorname{var}\left(\bar{Y}\right)} = \hat{\sigma}^2 / N \qquad \text{(C.8)}$$

$$\operatorname{se}\left(\bar{Y}\right) = \sqrt{\widehat{\operatorname{var}\left(\bar{Y}\right)}} = \hat{\sigma} / \sqrt{N} \qquad \text{(C.9)}$$

Principles of Econometrics, 3rd Edition *Slide C-20*

C.4.2 Estimating higher moments

$$\mu_r = E\left[\left(Y-\mu\right)^r\right]$$

In statistics the **Law of Large Numbers** says that sample means converge to population averages (expected values) as the sample size $N \to \infty$.

$$\tilde{\mu}_2 = \sum\left(Y_i - \bar{Y}\right)^2 \Big/ N = \tilde{\sigma}^2$$

$$\tilde{\mu}_3 = \sum\left(Y_i - \bar{Y}\right)^3 \Big/ N$$

$$\tilde{\mu}_4 = \sum\left(Y_i - \bar{Y}\right)^4 \Big/ N$$

Principles of Econometrics, 3rd Edition *Slide C-21*

C.4.2 Estimating higher moments

$$\widehat{skewness} = S = \frac{\tilde{\mu}_3}{\tilde{\sigma}^3}$$

$$\widehat{kurtosis} = K = \frac{\tilde{\mu}_4}{\tilde{\sigma}^4}$$

C.4.3 The hip data

$$\hat{\sigma}^2 = \frac{\sum (y_i - \bar{y})^2}{N-1} = \frac{\sum (y_i - 17.1582)^2}{49} = \frac{159.9995}{49} = 3.2653$$

$$\widehat{\text{var}}(\bar{Y}) = \frac{\hat{\sigma}^2}{N} = \frac{3.2653}{50} = .0653$$

$$\text{se}(\bar{Y}) = \hat{\sigma} \big/ \sqrt{N} = .2556$$

C.4.3 The hip data

$$\tilde{\sigma} = \sqrt{\sum (Y_i - \bar{Y})^2 \big/ N} = \sqrt{159.9995/50} = 1.7889$$

$$\tilde{\mu}_3 = \sum (Y_i - \bar{Y})^3 \big/ N = -.0791$$

$$\tilde{\mu}_4 = \sum (Y_i - \bar{Y})^4 \big/ N = 23.8748$$

C.4.4 Using the Estimates

$$P(Y>18)=P\left(\frac{Y-\mu}{\sigma}>\frac{18-\mu}{\sigma}\right)$$

$$\widehat{P(Y>18)}\cong P\left(\frac{Y-\bar{y}}{\hat{\sigma}}>\frac{18-17.158}{1.8070}\right)=P(Z>.4659)=\ .3207$$

$$\widehat{P(Y\le y^*)}\cong P\left(\frac{Y-\bar{y}}{\hat{\sigma}}\le\frac{y^*-17.1582}{1.8070}\right)=P\left(Z\le\frac{y^*-17.1582}{1.8070}\right)=\ .95$$

$$\frac{y^*-17.1582}{1.8070}=1.645\Rightarrow y^*=20.1305$$

C.5 Interval Estimation

- **C.5.1 Interval Estimation: σ^2 Known**

$$\bar{Y}=\sum_{i=1}^{N}Y_i/N$$

$$\bar{Y}\sim N(\mu,\sigma^2/N)$$

$$Z=\frac{\bar{Y}-\mu}{\sqrt{\sigma^2/N}}=\frac{\bar{Y}-\mu}{\sigma/\sqrt{N}}\sim N(0,1) \qquad \text{(C.10)}$$

$$P[Z\le z]=\Phi(z)$$

C.5.1 Interval Estimation: σ^2 Known

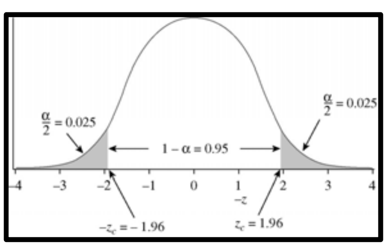

Figure C.4 Critical Values for the N(0,1) Distribution

C.5.1 Interval Estimation: σ^2 Known

$$P\left[Z \geq 1.96\right] = P\left[Z \leq -1.96\right] = .025$$

$$P\left[-1.96 \leq Z \leq 1.96\right] = 1 - .05 = .95 \quad \text{(C.11)}$$

$$P\left[\bar{Y} - 1.96\sigma/\sqrt{N} \leq \mu \leq \bar{Y} + 1.96\sigma/\sqrt{N}\right] = .95$$

Principles of Econometrics, 3rd Edition *Slide C-28*

C.5.1 Interval Estimation: σ^2 Known

$$P\left[\bar{Y} - z_c \frac{\sigma}{\sqrt{N}} \leq \mu \leq \bar{Y} + z_c \frac{\sigma}{\sqrt{N}}\right] = 1 - \alpha \quad \text{(C.12)}$$

$$\bar{Y} \pm z_c \frac{\sigma}{\sqrt{N}} \quad \text{(C.13)}$$

Principles of Econometrics, 3rd Edition *Slide C-29*

C.5.2 A Simulation

Table C.3 **30 Values from $N(10, 10)$**

11.939	11.407	13.809
10.706	12.157	7.443
6.644	10.829	8.855
13.187	12.368	9.461
8.433	10.052	2.439
9.210	5.036	5.527
7.961	14.799	9.921
14.921	10.478	11.814
6.223	13.859	13.403
10.123	12.355	10.819

Principles of Econometrics, 3rd Edition *Slide C-30*

C.5.2 A Simulation

Table C.4 Confidence Interval Estimates from 10 Samples of Data			
Sample	$\bar{y}$	Lower bound	Upper bound
1	10.206	9.074	11.338
2	9.828	8.696	10.959
3	11.194	10.063	12.326
4	8.822	7.690	9.953
5	10.434	9.303	11.566
6	8.855	7.723	9.986
7	10.511	9.380	11.643
8	9.212	8.080	10.343
9	10.464	9.333	11.596
10	10.142	9.010	11.273

C.5.2 A Simulation

- Any one interval estimate may or may not contain the true population parameter value.
- If *many* samples of size *N* are obtained, and intervals are constructed using (C.13) with $(1-\alpha) = .95$, then 95% of them will contain the true parameter value.
- A 95% level of "confidence" is the probability that the interval estimator will provide an interval containing the true parameter value. Our confidence is in the procedure, not in any one interval estimate.

C.5.3 Interval Estimation: σ^2 Unknown

- When σ^2 is unknown it is natural to replace it with its estimator $\hat{\sigma}^2$.

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{N}\left(Y_i - \bar{Y}\right)^2}{N-1}$$

$$t = \frac{\bar{Y} - \mu}{\hat{\sigma}/\sqrt{N}} \sim t_{(N-1)} \qquad \text{(C.14)}$$

C.5.3 Interval Estimation: σ² Unknown

$$P\left[-t_c \le \frac{\bar{Y}-\mu}{\hat{\sigma}/\sqrt{N}} \le t_c\right] = 1-\alpha$$

$$P\left[\bar{Y} - t_c\frac{\hat{\sigma}}{\sqrt{N}} \le \mu \le \bar{Y} + t_c\frac{\hat{\sigma}}{\sqrt{N}}\right] = 1-\alpha$$

$$\bar{Y} \pm t_c\frac{\hat{\sigma}}{\sqrt{N}} \text{ or } \bar{Y} \pm t_c \text{se}\left(\bar{Y}\right) \quad \text{(C.15)}$$

Principles of Econometrics, 3rd Edition *Slide C-34*

C.5.3 Interval Estimation: σ² Unknown

Remark: The confidence interval (C.15) is based upon the assumption that the population is normally distributed, so that $\bar{Y}$ is normally distributed. If the population is not normal, then we invoke the central limit theorem, and say that $\bar{Y}$ is approximately normal in "large" samples, which from Figure C.3 you can see might be as few as 30 observations. In this case we can use (C.15), recognizing that there is an approximation error introduced in smaller samples.

Principles of Econometrics, 3rd Edition *Slide C-35*

C.5.4 A Simulation (continued)

Table C.5 **Interval Estimates Using (C.15) from 10 Samples**

Sample	$\bar{y}$	$\hat{\sigma}^2$	Lower bound	Upper bound
1	10.206	9.199	9.073	11.338
2	9.828	6.876	8.849	10.807
3	11.194	10.330	9.994	12.394
4	8.822	9.867	7.649	9.995
5	10.434	7.985	9.379	11.489
6	8.855	6.230	7.923	9.787
7	10.511	7.333	9.500	11.523
8	9.212	14.687	7.781	10.643
9	10.464	10.414	9.259	11.669
10	10.142	17.689	8.571	11.712

Principles of Econometrics, 3rd Edition *Slide C-36*

C.5.5 Interval estimation using the hip data

Given a random sample of size N = 50 we estimated the mean U.S. hip width to be = 17.158 inches.

$$\hat{\sigma}^2 = 3.265 \text{ therefore } \hat{\sigma} = 1.807$$

$$\hat{\sigma}\big/\sqrt{N} = 1.807\big/\sqrt{50} = .2556$$

$$\bar{y} \pm t_c \frac{\hat{\sigma}}{\sqrt{N}} = 17.1582 \pm 2.01 \frac{1.807}{\sqrt{50}} = [16.6447,\ 17.6717]$$

C.6 Hypothesis Tests About A Population Mean

Components of Hypothesis Tests

A *null* hypothesis, H_0

An *alternative* hypothesis, H_1

A test *statistic*

A *rejection* region

A conclusion

C.6.1 Components of Hypothesis Tests

- ***The Null Hypothesis***

The "null" hypothesis, which is denoted H_0 (*H-naught*), specifies a value c for a parameter. We write the null hypothesis as $H_0 : \mu = c.$ A null hypothesis is the belief we will maintain until we are convinced by the sample evidence that it is not true, in which case we *reject* the null hypothesis.

C.6.1 Components of Hypothesis Tests

- ***The Alternative Hypothesis***
  - H_1: $\mu > c$ If we reject the null hypothesis that $\mu = c$, we accept the alternative that μ is greater than c.
  - H_1: $\mu < c$ If we reject the null hypothesis that $\mu = c$, we accept the alternative that μ is less than c.
  - H_1: $\mu \neq c$ If we reject the null hypothesis that $\mu = c$, we accept the alternative that μ takes a value other than (not equal to) c.

Principles of Econometrics, 3rd Edition *Slide C-40*

C.6.1 Components of Hypothesis Tests

- ***The Test Statistic***

A test statistic's probability distribution is completely known when the null hypothesis is true, and it has some other distribution if the null hypothesis is not true.

$$t = \frac{\bar{Y} - \mu}{\hat{\sigma}/\sqrt{N}} \sim t_{(N-1)} \qquad \text{If } H_0 : \mu = c \text{ is true then}$$

$$t = \frac{\bar{Y} - c}{\hat{\sigma}/\sqrt{N}} \sim t_{(N-1)} \qquad \text{(C.16)}$$

Principles of Econometrics, 3rd Edition *Slide C-41*

C.6.1 Components of Hypothesis Tests

Remark: The test statistic distribution in (C.16) is based on an assumption that the population is normally distributed. If the population is not normal, then we invoke the central limit theorem, and say that $\bar{Y}$ is approximately normal in "large" samples. We can use (C.16), recognizing that there is an approximation error introduced if our sample is small.

Principles of Econometrics, 3rd Edition *Slide C-42*

C.6.1 Components of Hypothesis Tests

- *The Rejection Region*
  - If a value of the test statistic is obtained that falls in a region of low probability, then it is unlikely that the test statistic has the assumed distribution, and thus it is unlikely that the null hypothesis is true.
  - If the alternative hypothesis is true, then values of the test statistic will tend to be unusually "large" or unusually "small", determined by choosing a probability α, called the **level of significance** of the test.
  - The level of significance of the test α is usually chosen to be .01, .05 or .10.

Principles of Econometrics, 3rd Edition *Slide C-43*

C.6.1 Components of Hypothesis Tests

- *A Conclusion*
  - When you have completed a hypothesis test you should state your conclusion, whether you reject, or do not reject, the null hypothesis.
  - Say what the conclusion means in the economic context of the problem you are working on, i.e., interpret the results in a meaningful way.

Principles of Econometrics, 3rd Edition *Slide C-44*

C.6.2 One-tail Tests with Alternative "Greater Than" (>)

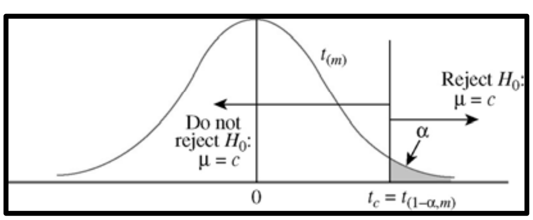

Figure C.5 The rejection region for the one-tail test of H_1: $\mu = c$ against H_1: $\mu > c$

Principles of Econometrics, 3rd Edition *Slide C-45*

C.6.3 One-tail Tests with Alternative "Less Than" (<)

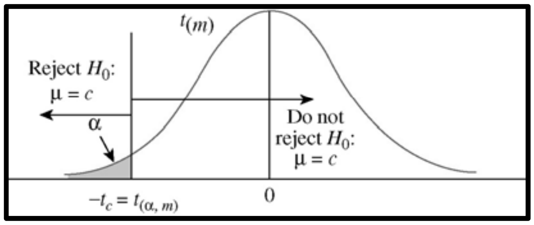

Figure C.6 The rejection region for the one-tail test of H_1: $\mu = c$ against H_1: $\mu < c$

C.6.4 Two-tail Tests with Alternative "Not Equal To" (≠)

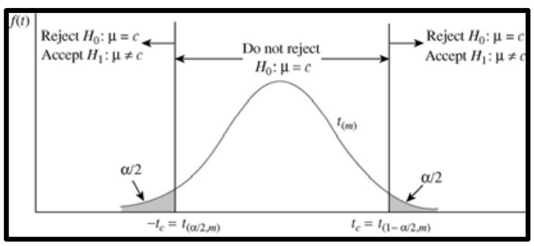

Figure C.7 The rejection region for a test of H_1: $\mu = c$ against H_1: $\mu \neq c$

C.6.5 Example of a One-tail Test Using the Hip Data

- The null hypothesis is $H_0: \mu = 16.5$.
  The alternative hypothesis is $H_1: \mu > 16.5$.
- The test statistic $t = \dfrac{\bar{Y} - 16.5}{\hat{\sigma}/\sqrt{N}} \sim t_{(N-1)}$ *if the null hypothesis is true.*
- The level of significance α=.05.
  $t_c = t_{(.95,49)} = 1.6766$

C.6.5 Example of a One-tail Test Using the Hip Data

- The value of the test statistic is

$$t = \frac{17.1582 - 16.5}{1.807 / \sqrt{50}} = 2.5756.$$

- Conclusion: Since $t = 2.5756 > 1.68$ we *reject* the null hypothesis. The sample information we have is *incompatible* with the hypothesis that $\mu = 16.5$. We accept the alternative that the population mean hip size is greater than 16.5 inches, at the α=.05 level of significance.

Principles of Econometrics, 3rd Edition *Slide C-49*

C.6.6 Example of a Two-tail Test Using the Hip Data

- The null hypothesis is $H_0 : \mu = 17.$
  The alternative hypothesis is $H_1 : \mu \neq 17.$

- The test statistic $t = \frac{\bar{Y} - 17}{\hat{\sigma} / \sqrt{N}} \sim t_{(N-1)}$ *if the null hypothesis is true.*

- The level of significance α=.05, therefore $\alpha/2 = .025.$
  $t_c = t_{(.975,49)} = 2.01$

Principles of Econometrics, 3rd Edition *Slide C-50*

C.6.6 Example of a Two-tail Test Using the Hip Data

- The value of the test statistic is

$$t = \frac{17.1582 - 17}{1.807 / \sqrt{50}} = .6191.$$

- Conclusion: Since $-2.01 < t = .6191 < 2.01$ we *do not reject* the null hypothesis. The sample information we have is *compatible* with the hypothesis that the population mean hip size $\mu = 17$.

Principles of Econometrics, 3rd Edition *Slide C-51*

C.6.6 Example of a Two-tail Test Using the Hip Data

Warning: Care must be taken here in interpreting the outcome of a statistical test. One of the basic precepts of hypothesis testing is that finding a sample value of the test statistic in the non-rejection region does not make the null hypothesis true! The weaker statements "we do not reject the null hypothesis," or "we fail to reject the null hypothesis," do not send a misleading message.

Principles of Econometrics, 3rd Edition *Slide C-52*

C.6.7 The *p*-value

***p*-value rule**: Reject the null hypothesis when the *p*-value is less than, or equal to, the level of significance α. That is, if $p \le \alpha$ then reject H_0. If $p > \alpha$ then do not reject H_0

Principles of Econometrics, 3rd Edition *Slide C-53*

C.6.7 The *p*-value

- How the *p*-value is computed depends on the alternative. If t is the calculated value [not the critical value t_c] of the t-statistic with $N-1$ degrees of freedom, then:
  - if H_1: $\mu > c$, p = probability to the right of t
  - if H_1: $\mu < c$, p = probability to the left of t
  - if H_1: $\mu \neq c$, p = <u>sum</u> of probabilities to the right of $|t|$ <u>and</u> to the left of $-|t|$

Principles of Econometrics, 3rd Edition *Slide C-54*

C.6.7 The *p*-value

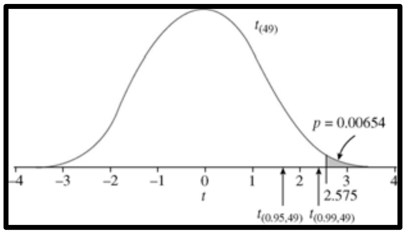

Figure C.8 The *p*-value for a right-tail test

C.6.7 The *p*-value

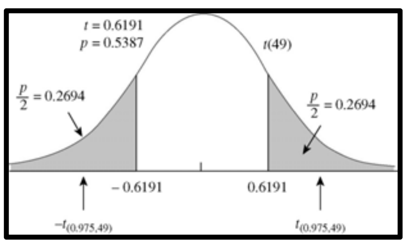

Figure C.9 The *p*-value for a two-tailed test

C.6.8 A Comment on Stating Null and Alternative Hypotheses

- A statistical test procedure cannot prove the truth of a null hypothesis. When we fail to reject a null hypothesis, all the hypothesis test can establish is that the information in a sample of data is *compatible* with the null hypothesis. On the other hand, a statistical test can lead us to *reject* the null hypothesis, with only a small probability, α, of rejecting the null hypothesis when it is actually true. Thus rejecting a null hypothesis is a stronger conclusion than failing to reject it.

C.6.9 Type I and Type II errors

Correct Decisions
The null hypothesis is *false* and we decide to *reject* it.
The null hypothesis is *true* and we decide *not* to reject it.

Incorrect Decisions
The null hypothesis is *true* and we decide to *reject* it (a Type I error)
The null hypothesis is *false* and we decide *not* to reject it (a Type II error)

C.6.9 Type I and Type II errors

- The probability of a Type II error varies inversely with the level of significance of the test, α, which is the probability of a Type I error. If you choose to make α smaller, the probability of a Type II error increases.
- If the null hypothesis is $\mu = c$, and if the true (unknown) value of μ is *close* to c, then the probability of a Type II error is high.
- The larger the sample size N, the lower the probability of a Type II error, given a level of Type I error α.

C.6.10 A Relationship Between Hypothesis Testing and Confidence Intervals

$$H_0 : \mu = c$$
$$H_1 : \mu \neq c$$

- If we fail to reject the null hypothesis at the α level of significance, then the value c will fall within a $100(1-\alpha)\%$ confidence interval estimate of μ.
- If we reject the null hypothesis, then c will fall outside the $100(1-\alpha)\%$ confidence interval estimate of μ.

C.6.10 A Relationship Between Hypothesis Testing and Confidence Intervals

- We fail to reject the null hypothesis when $-t_c \le t \le t_c$, or when

$$-t_c \le \frac{\bar{Y}-c}{\hat{\sigma}/\sqrt{N}} \le t_c$$

$$\bar{Y} - t_c \frac{\hat{\sigma}}{\sqrt{N}} \le c \le \bar{Y} + t_c \frac{\hat{\sigma}}{\sqrt{N}}$$

Principles of Econometrics, 3rd Edition *Slide C-61*

C.7 Some Useful Tests

- **C.7.1 Testing the population variance**

$$Y \sim N\left(\mu,\sigma^2\right),\ \bar{Y} = \sum Y_i \Big/ N$$

$$\hat{\sigma}^2 = \sum\left(Y_i - \bar{Y}\right)^2 \Big/ \left(N-1\right)$$

$$H_0 : \sigma^2 = \sigma_0^2$$

$$V = \frac{(N-1)\hat{\sigma}^2}{\sigma_0^2} \sim \chi^2_{(N-1)}$$

Principles of Econometrics, 3rd Edition *Slide C-62*

C.7.1 Testing the Population Variance

If $H_1 : \sigma^2 > \sigma_0^2$, then the null hypothesis is rejected if $V \ge \chi^2_{(.95,N-1)}$.

If $H_1 : \sigma^2 \ne \sigma_0^2$, then we carry out a two – tail test, and the null hypothesis is rejected if $V \ge \chi^2_{(.975,N-1)}$ or if $V \le \chi^2_{(.025,N-1)}$.

Principles of Econometrics, 3rd Edition *Slide C-63*

C.7.2 Testing the Equality of two Population Means

Case 1: Population variances are equal

$$\sigma_1^2 = \sigma_2^2 = \sigma_p^2$$

$$\hat{\sigma}_p^2 = \frac{(N_1 - 1)\hat{\sigma}_1^2 + (N_2 - 1)\hat{\sigma}_2^2}{N_1 + N_2 - 2}$$

If the null hypothesis $H_0 : \mu_1 - \mu_2 = c$ is true then

$$t = \frac{(\bar{Y}_1 - \bar{Y}_2) - c}{\sqrt{\hat{\sigma}_p^2 \left(\frac{1}{N_1} + \frac{1}{N_2} \right)}} \sim t_{(N_1 + N_2 - 2)}$$

Principles of Econometrics, 3rd Edition *Slide C-64*

C.7.2 Testing the Equality of two Population Means

Case 2: Population variances are unequal

$$t^* = \frac{(\bar{Y}_1 - \bar{Y}_2) - c}{\sqrt{\frac{\hat{\sigma}_1^2}{N_1} + \frac{\hat{\sigma}_2^2}{N_2}}}$$

$$df = \frac{\left(\hat{\sigma}_1^2/N_1 + \hat{\sigma}_2^2/N_2\right)^2}{\left(\frac{\left(\hat{\sigma}_1^2/N_1\right)^2}{N_1 - 1} + \frac{\left(\hat{\sigma}_2^2/N_2\right)^2}{N_2 - 1} \right)}$$

Principles of Econometrics, 3rd Edition *Slide C-65*

C.7.3 Testing the ratio of two population variances

$$F = \frac{\dfrac{(N_1 - 1)\hat{\sigma}_1^2/\sigma_1^2}{(N_1 - 1)}}{\dfrac{(N_2 - 1)\hat{\sigma}_2^2/\sigma_2^2}{(N_2 - 1)}} = \frac{\hat{\sigma}_1^2/\sigma_1^2}{\hat{\sigma}_2^2/\sigma_2^2} \sim F_{(N_1 - 1, N_2 - 1)}$$

Principles of Econometrics, 3rd Edition *Slide C-66*

C.7.4 Testing the normality of a population

The normal distribution is symmetric, and has a bell-shape with a peakedness and tail-thickness leading to a kurtosis of 3. We can test for departures from normality by checking the skewness and kurtosis from a sample of data.

$$\widehat{skewness} = S = \frac{\tilde{\mu}_3}{\tilde{\sigma}^3}$$

$$\widehat{kurtosis} = K = \frac{\tilde{\mu}_4}{\tilde{\sigma}^4}$$

Principles of Econometrics, 3rd Edition *Slide C-67*

C.7.4 Testing the normality of a population

The **Jarque-Bera** test statistic allows a joint test of these two characteristics,

$$JB = \frac{N}{6}\left(S^2 + \frac{(K-3)^2}{4} \right)$$

If we reject the null hypothesis then we know the data have non-normal characteristics, but we do not know what distribution the population might have.

Principles of Econometrics, 3rd Edition *Slide C-68*

C.7.4 Testing the normality of a population

For the Hip data,

$$JB = \frac{N}{6}\left(S^2 + \frac{(K-3)^2}{4} \right) = \frac{50}{6}\left((-.0138)^2 + \frac{(2.3315-3)^2}{4} \right) = .9325$$

$$p = P\left[\chi^2_{(2)} \geq .9325\right] = .6273$$

Principles of Econometrics, 3rd Edition *Slide C-69*

C.8 Introduction to Maximum Likelihood Estimation

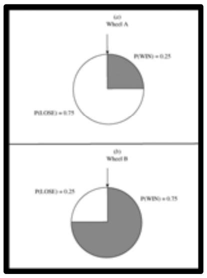

Figure C.10 Wheel of Fortune Game

C.8 Introduction to Maximum Likelihood Estimation

- For wheel *A*, with *p*=1/4, the probability of observing WIN, WIN, LOSS is

$$\frac{1}{4}\times\frac{1}{4}\times\frac{3}{4}=\frac{3}{64}=.0469$$

- For wheel *B*, with *p*=3/4, the probability of observing WIN, WIN, LOSS is

$$\frac{3}{4}\times\frac{3}{4}\times\frac{1}{4}=\frac{9}{64}=.1406$$

C.8 Introduction to Maximum Likelihood Estimation

- If we had to choose wheel *A* or *B* based on the available data, we would choose wheel *B* because it has a higher probability of having produced the observed data.
- It is more *likely* that wheel *B* was spun than wheel *A*, and $\hat{p}=3/4$ is called the **maximum likelihood estimate** of *p*.
- The **maximum likelihood principle** seeks the parameter values that maximize the probability, or likelihood, of observing the outcomes actually obtained.

C.8 Introduction to Maximum Likelihood Estimation

Suppose p can be any probability between zero and one. The probability of observing WIN, WIN, LOSS is the likelihood L, and is

$$L(p) = p \times p \times (1-p) = p^2 - p^3 \qquad \text{(C.17)}$$

We would like to find the value of p that maximizes the likelihood of observing the outcomes actually obtained.

C.8 Introduction to Maximum Likelihood Estimation

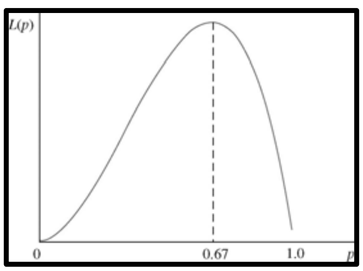

Figure C.11 A Likelihood Function

C.8 Introduction to Maximum Likelihood Estimation

$$\frac{dL(p)}{dp} = 2p - 3p^2$$

$$2p - 3p^2 = 0 \Rightarrow p(2-3p) = 0$$

There are two solutions to this equation, p=0 or p=2/3.

The value that maximizes $L(p)$ is $\hat{p} = 2/3$, which is the maximum likelihood estimate.

C.8 Introduction to Maximum Likelihood Estimation

Let us define the random variable X that takes the values x=1 (WIN) and x=0 (LOSS) with probabilities p and 1−p.

$$P[X=x]=f(x\mid p)=p^x(1-p)^{1-x}, \quad x=0,1$$

$$\begin{aligned} f(x_1,\ldots,x_N\mid p) &= f(x_1\mid p)\times\cdots\times f(x_N\mid p) \\ &= p^{\sum x_i}(1-p)^{N-\sum x_i} \\ &= L(p\mid x_1,\ldots,x_N) \end{aligned} \quad \text{(C.18)}$$

C.8 Introduction to Maximum Likelihood Estimation

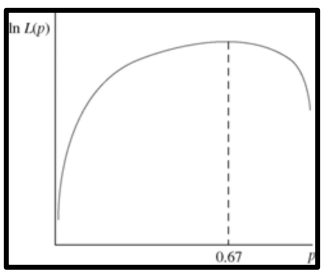

Figure C.12 A Log-Likelihood Function

C.8 Introduction to Maximum Likelihood Estimation

$$\begin{aligned} \ln L(p) &= \sum_{i=1}^{N}\ln f(x_i\mid p) \\ &= \left(\sum_{i=1}^{N}x_i\right)\ln(p)+\left(N-\sum_{i=1}^{N}x_i\right)\ln(1-p) \end{aligned} \quad \text{(C.19)}$$

$$\frac{d\ln L(p)}{dp}=\frac{\sum x_i}{p}-\frac{N-\sum x_i}{1-p}$$

C.8 Introduction to Maximum Likelihood Estimation

$$\frac{\sum x_i}{\hat{p}} - \frac{N - \sum x_i}{1 - \hat{p}} = 0$$

$$(1 - \hat{p})\sum x_i - \hat{p}\left(N - \sum x_i\right) = 0$$

$$\hat{p} = \frac{\sum x_i}{N} = \bar{x} \qquad \text{(C.20)}$$

C.8.1 Inference with Maximum Likelihood Estimators

$$\ln L(\theta) = \sum_{i=1}^{N} \ln f(x_i \mid \theta)$$

$$\hat{\theta} \overset{a}{\sim} N(\theta, V) \qquad \text{(C.21)}$$

$$t = \frac{\hat{\theta} - c}{\text{se}(\hat{\theta})} \overset{a}{\sim} t_{(N-1)} \qquad \text{(C.22)}$$

C.8.1 Inference with Maximum Likelihood Estimators

REMARK: The asymptotic results in (C.21) and (C.22) hold only in large samples. The distribution of the test statistic can be approximated by a *t*-distribution with $N-1$ degrees of freedom. If N is truly large then the $t_{(N-1)}$ distribution converges to the standard normal distribution $N(0,1)$. When the sample size N may not be large, we prefer using the *t*-distribution critical values, which are adjusted for small samples by the degrees of freedom correction, when obtaining interval estimates and carrying out hypothesis tests.

C.8.2 The Variance of the Maximum Likelihood Estimator

$$V = \text{var}(\hat{\theta}) = \left[-E\left(\frac{d^2 \ln L(\theta)}{d\theta^2} \right) \right]^{-1} \quad \text{(C.23)}$$

C.8.2 The Variance of the Maximum Likelihood Estimator

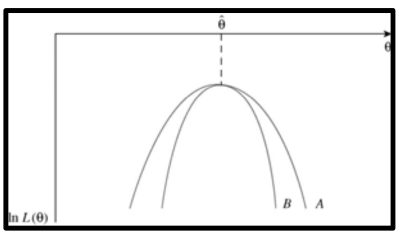

Figure C.13 Two Log-Likelihood Functions

C.8.3 The Distribution of the Sample Proportion

$$\frac{d^2 \ln L(p)}{dp^2} = -\frac{\sum x_i}{p^2} - \frac{N - \sum x_i}{(1-p)^2} \quad \text{(C.24)}$$

$$E(x_i) = 1 \times P(x_i = 1) + 0 \times P(x_i = 0) = 1 \times p + 0 \times (1-p) = p$$

C.8.3 The Distribution of the Sample Proportion

$$E\left(\frac{d^2 \ln L(p)}{dp^2}\right) = -\frac{\sum E(x_i)}{p^2} - \frac{N - \sum E(x_i)}{(1-p)^2}$$

$$= -\frac{Np}{p^2} - \frac{N - Np}{(1-p)^2}$$

$$= -\frac{N}{p(1-p)}$$

C.8.3 The Distribution of the Sample Proportion

$$V = \text{var}(\hat{p}) = \left[-E\left(\frac{d^2 \ln L(p)}{dp^2}\right)\right]^{-1} = \frac{p(1-p)}{N}$$

$$\hat{p} \overset{a}{\sim} N\left(p, \frac{p(1-p)}{N}\right)$$

C.8.3 The Distribution of the Sample Proportion

$$\hat{V} = \frac{\hat{p}(1-\hat{p})}{N}$$

$$\text{se}(\hat{p}) = \sqrt{\hat{V}} = \sqrt{\frac{\hat{p}(1-\hat{p})}{N}}$$

C.8.3 The Distribution of the Sample Proportion

$$\mathrm{se}(\hat{p}) = \sqrt{\frac{\hat{p}(1-\hat{p})}{N}} = \sqrt{\frac{.375 \times .625}{200}} = .0342$$

$$t = \frac{\hat{p} - .4}{\mathrm{se}(\hat{p})} = \frac{.375 - .4}{.0342} = -.7303$$

$$\hat{p} \pm 1.96\,\mathrm{se}(\hat{p}) = .375 \pm 1.96(.0342) = [.3075, .4425]$$

C.8.4 Asymptotic Test Procedures

C.8.4a The likelihood ratio (LR) test

The **likelihood ratio statistic** which is twice the difference between $\ln L(\hat{\theta})$ and $\ln L(c)$.

$$LR = 2\left[\ln L(\hat{\theta}) - \ln L(c)\right] \quad \text{(C.25)}$$

C.8.4a The likelihood ratio (LR) test

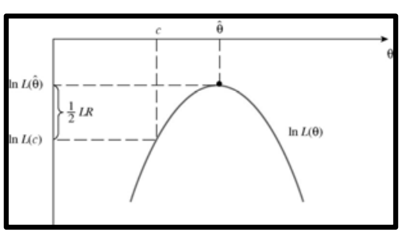

Figure C.14 The Likelihood Ratio Test

C.8.4a The likelihood ratio (LR) test

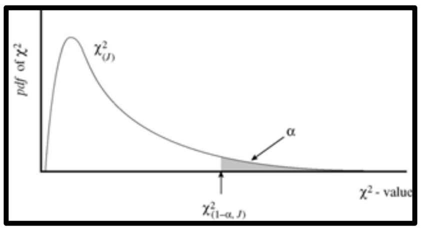

Figure C.15 Critical Value for a Chi-Square Distribution

C.8.4a The likelihood ratio (LR) test

$$\ln L(\hat{p}) = \left(\sum_{i=1}^{N} x_i\right)\ln \hat{p} + \left(N - \sum_{i=1}^{N} x_i\right)\ln(1-\hat{p})$$

$$= N\hat{p}\ln \hat{p} + \left(N - N\hat{p}\right)\ln(1-\hat{p})$$

$$= N\left[\hat{p}\ln \hat{p} + \left(1-\hat{p}\right)\ln(1-\hat{p})\right]$$

C.8.4a The likelihood ratio (LR) test

For the cereal box problem $\hat{p} = .375$ and $N = 200$.

$$\ln L(\hat{p}) = 200\left[.375 \times \ln(.375) + (1-.375)\ln(1-.375)\right]$$

$$= -132.3126$$

C.8.4a The likelihood ratio (LR) test

The value of the log-likelihood function assuming $H_0 : p = .4$ is true is:

$$\ln L(.4) = \left(\sum_{i=1}^{N} x_i\right)\ln(.4) + \left(N - \sum_{i=1}^{N} x_i\right)\ln(1-.4)$$

$$= 75 \times \ln(.4) + (200 - 75) \times \ln(.6)$$

$$= -132.5750$$

C.8.4a The likelihood ratio (LR) test

The problem is to assess whether −132.3126 is significantly different from −132.5750.

The *LR* test statistic (C.25) is:

$$LR = 2\,[\ln L(\hat{p}) - \ln L(.4)] = 2 \times \left(-132.3126 - (-132.575)\right) = .5247$$

The critical value is $\chi^2_{(.95,1)} = 3.84$.

Since .5247 < 3.84 we do not reject the null hypothesis.

C.8.4b The Wald test

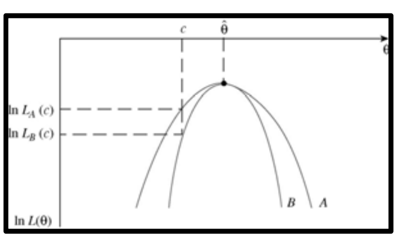

Figure C.16 The Wald Statistic

C.8.4b The Wald test

$$W = \left(\hat{\theta} - c\right)^2 \left[-\frac{d^2 \ln L(\theta)}{d\theta^2} \right] \tag{C.26}$$

If the null hypothesis is true then the Wald statistic (C.26) has a $\chi^2_{(1)}$ distribution, and we reject the null hypothesis if $W \geq \chi^2_{(1-\alpha,1)}$.

Principles of Econometrics, 3rd Edition *Slide C-97*

C.8.4b The Wald test

$$I(\theta) = -E\left[\frac{d^2 \ln L(\theta)}{d\theta^2} \right] = V^{-1} \tag{C.27}$$

$$W = \left(\hat{\theta} - c\right)^2 I(\theta) \tag{C.28}$$

$$W = \left(\hat{\theta} - c\right)^2 V^{-1} = \left(\hat{\theta} - c\right)^2 \big/ V \tag{C.29}$$

Principles of Econometrics, 3rd Edition *Slide C-98*

C.8.4b The Wald test

$$\hat{V} = \left[I\left(\hat{\theta}\right) \right]^{-1} \tag{C.30}$$

$$\sqrt{W} = \frac{\hat{\theta} - c}{\sqrt{\hat{V}}} = \frac{\hat{\theta} - c}{\mathrm{se}\left(\hat{\theta}\right)} = t$$

Principles of Econometrics, 3rd Edition *Slide C-99*

C.8.4b The Wald test

In the blue box-green box example:

$$I(\hat{p}) = \hat{V}^{-1} = \frac{N}{\hat{p}(1-\hat{p})} = \frac{200}{.375(1-.375)} = 853.3333$$

$$W = (\hat{p}-c)^2 I(\hat{p}) = (.375-.4)^2 \times 853.3333 = .5333$$

C.8.4c The Lagrange multiplier (LM) test

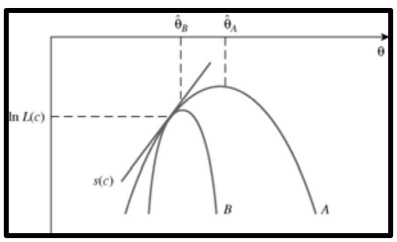

Figure C.17 Motivating the Lagrange multiplier test

C.8.4c The Lagrange multiplier (LM) test

$$s(\theta) = \frac{d \ln L(\theta)}{d\theta} \quad \text{(C.31)}$$

$$LM = \frac{[s(c)]^2}{I(\theta)} = [s(c)]^2 [I(\theta)]^{-1} \quad \text{(C.32)}$$

$$LM = [s(c)]^2 [I(c)]^{-1}$$

$$W = (\hat{\theta}-c)^2 I(\hat{\theta})$$

C.8.4c The Lagrange multiplier (LM) test

In the blue box-green box example:

$$s(.4)=\frac{\sum x_i}{c}-\frac{N-\sum x_i}{1-c}=\frac{75}{.4}-\frac{200-75}{1-.4}=-20.8333$$

$$I(.4)=\frac{N}{c(1-c)}=\frac{200}{.4(1-.4)}=833.3333$$

$$LM=\left[s(.4)\right]^2\left[I(.4)\right]^{-1}=\left[-20.8333\right]^2\left[833.3333\right]^{-1}=.5208$$

C.9 Algebraic Supplements

- **C.9.1 Derivation of Least Squares Estimator**

$$S=\sum_{i=1}^{N}(y_i-\mu)^2$$

$$d_i=\sqrt{(y_i-\mu)^2}$$

$$d_i^2=(y_i-\mu)^2$$

C.9.1 Derivation of Least Squares Estimator

$$S(\mu)=\sum_{i=1}^{N}d_i^2=\sum_{i=1}^{N}(y_i-\mu)^2$$

$$S(\mu)=\sum_{i=1}^{N}y_i^2-2\mu\sum_{i=1}^{N}y_i+N\mu^2 \quad =a_0-2a_1\mu+a_2\mu^2$$

$$a_0=\sum y_i^2=14880.1909,\quad a_1=\sum y_i=857.9100,\quad a_2=N=50$$

C.9.1 Derivation of Least Squares Estimator

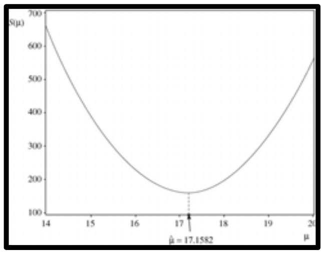

Figure C.18 The Sum of Squares Parabola For the Hip Data

C.9.1 Derivation of Least Squares Estimator

$$\frac{dS(\mu)}{d\mu} = -2a_1 + 2a_2\mu$$

$$-2a_1 + 2a_2\hat{\mu} = 0$$

$$\hat{\mu} = \frac{a_1}{a_2} = \frac{\sum_{i=1}^{N} y_i}{N} = \bar{y}$$

$$\hat{\mu} = \frac{\sum_{i=1}^{N} Y_i}{N} = \bar{Y}$$

C.9.1 Derivation of Least Squares Estimator

For the hip data in Table C.1

$$\hat{\mu} = \frac{\sum_{i=1}^{N} y_i}{N} = \frac{857.9100}{50} = 17.1582$$

Thus we estimate that the average hip size in the population is 17.1582 inches.

C.9.2 Best Linear Unbiased Estimation

$$\bar{Y} = \sum_{i=1}^{N} Y_i / N = \frac{1}{N}Y_1 + \frac{1}{N}Y_2 + \ldots + \frac{1}{N}Y_N$$

$$= a_1 Y_1 + a_2 Y_2 + \ldots + a_N Y_N$$

$$= \sum_{i=1}^{N} a_i Y_i$$

Principles of Econometrics, 3rd Edition *Slide C-109*

C.9.2 Best Linear Unbiased Estimation

$$\tilde{Y} = \sum_{i=1}^{N} a_i^* Y_i$$

$$a_i^* = a_i + c_i = \frac{1}{N} + c_i$$

Principles of Econometrics, 3rd Edition *Slide C-110*

C.9.2 Best Linear Unbiased Estimation

$$\tilde{Y} = \sum_{i=1}^{N} a_i^* Y_i = \sum_{i=1}^{N} \left(\frac{1}{N} + c_i \right) Y_i$$

$$= \sum_{i=1}^{N} \frac{1}{N} Y_i + \sum_{i=1}^{N} c_i Y_i$$

$$= \bar{Y} + \sum_{i=1}^{N} c_i Y_i$$

Principles of Econometrics, 3rd Edition *Slide C-111*

C.9.2 Best Linear Unbiased Estimation

$$E\left[\tilde{Y}\right] = E\left[\bar{Y} + \sum_{i=1}^{N} c_i Y_i\right] = \mu + \sum_{i=1}^{N} c_i E\left[Y_i\right]$$

$$= \mu + \mu \sum_{i=1}^{N} c_i$$

Principles of Econometrics, 3rd Edition Slide C-112

C.9.2 Best Linear Unbiased Estimation

$$\text{var}(\tilde{Y}) = \text{var}\left(\sum_{i=1}^{N} a_i^* Y_i\right) = \text{var}\left(\sum_{i=1}^{N}\left(\frac{1}{N} + c_i\right) Y_i\right) = \sum_{i=1}^{N}\left(\frac{1}{N} + c_i\right)^2 \text{var}(Y_i)$$

$$= \sigma^2 \sum_{i=1}^{N}\left(\frac{1}{N} + c_i\right)^2 = \sigma^2 \sum_{i=1}^{N}\left(\frac{1}{N^2} + \frac{2}{N} c_i + c_i^2\right) = \sigma^2\left(\frac{1}{N} + \frac{2}{N}\sum_{i=1}^{N} c_i + \sum_{i=1}^{N} c_i^2\right)$$

$$= \sigma^2 / N + \sigma^2 \sum_{i=1}^{N} c_i^2 \qquad \left(\text{since } \sum_{i=1}^{N} c_i = 0\right)$$

$$= \text{var}(\bar{Y}) + \sigma^2 \sum_{i=1}^{N} c_i^2$$

Principles of Econometrics, 3rd Edition Slide C-113

Keywords

- alternative hypothesis
- asymptotic distribution
- BLUE
- central limit theorem
- central moments
- estimate
- estimator
- experimental design
- information measure
- interval estimate
- Lagrange multiplier test
- Law of large numbers
- level of significance
- likelihood function
- likelihood ratio test
- linear estimator
- log likelihood function
- maximum likelihood estimation
- null hypothesis
- point estimate
- population parameter
- *p*-value
- random sample
- rejection region
- sample mean
- sample variance
- sampling distribution
- sampling variation
- standard error
- standard error of the mean
- standard error of the estimate
- statistical inference
- test statistic
- two-tail tests
- Type I error
- Type II error
- unbiased estimators
- Wald test

Principles of Econometrics, 3rd Edition Slide C-114